Forces and stresses

Consitutive modelling is mainly about the relation between things like forces and things like displacements. In this chapter we examine the forces. In the next chapter we will discuss the displacements and their relationship with forces. In the third and last chapter, we examine some methods for solving problems.

1 Forces and momentum

1.1 Types of forces

We can distinguish two important types of forces. These are the **(distributed) contact forces t** and the **(distributed) mass forces b**. They are defined as

$$\mathbf{t} = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{F}}{\Delta A} \qquad \text{and} \qquad \mathbf{b} = \lim_{\Delta V \to 0} \frac{\Delta \mathbf{F}}{\Delta V}. \tag{1.1}$$

Here $\mathbf{F}$ denotes a force, A denotes an area and V denotes a volume. Now let's examine a certain volume Ω. The **total contact force $\mathbf{F_s}$** and the **total body force $\mathbf{F_b}$** can be found using

$$\mathbf{F_s} = \int_{\partial\Omega} \mathbf{t}(\mathbf{x})\, dA \qquad \text{and} \qquad \mathbf{F_b} = \int_{\Omega} \mathbf{b}(\mathbf{x})\, dV. \tag{1.2}$$

(The signal $\partial\Omega$ means we integrate over the surface of the volume Ω.) Together, the total contact force F_s and the total body force form the **total external force** F_{ext}.

1.2 Linear momentum

Again, we examine a volume Ω. The **total linear momentum** P of the volume can be found using

$$\mathbf{P} = \int_{\Omega} \rho \mathbf{v}\, dV, \tag{1.3}$$

where ρ denotes the density of the volume and $\mathbf{v}$ the velocity. It can be shown that $\mathbf{F_{ext}} = d\mathbf{P}/dt$. In other words,

$$\int_{\partial\Omega} \mathbf{t}\, dA + \int_{\Omega} \mathbf{b}\, dV = \frac{d}{dt} \int_{\Omega} \rho \mathbf{v}\, dV = \int_{\Omega} \rho \mathbf{a}\, dV. \tag{1.4}$$

1.3 Moments and angular momentum

Forces also cause moments. The **moment due to surface forces $\mathbf{M_s}$** and **the moment due to body forces $\mathbf{M_b}$** can be found using

$$\mathbf{M_s} = \int_{\partial\Omega} \mathbf{x} \times \mathbf{t}\, dA \qquad \text{and} \qquad M_b = \int_{\Omega} \mathbf{x} \times \mathbf{b}\, dV. \tag{1.5}$$

Together, these two moments form the **total moment of external forces** M_{ext}. (All moments are about the origin.)

We can also find the **total angular momentum** H. (Also with respect to the origin.) We do this using

$$\mathbf{H} = \int_{\Omega} \mathbf{x} \times \rho \mathbf{v}\, dV. \tag{1.6}$$

Similar to linear momentum, it also holds that $\mathbf{M_{ext}} = d\mathbf{H}/dt$. From this, it can be derived that

$$\int_{\partial\Omega} \mathbf{x} \times \mathbf{t}\, dA + \int_{\Omega} \mathbf{x} \times \mathbf{b}\, dV = \frac{d}{dt} \int_{\Omega} \mathbf{x} \times \rho\mathbf{v}\, dV = \int_{\Omega} \mathbf{x} \times \rho\mathbf{a}\, dV. \tag{1.7}$$

2 Stress vectors and tensors

2.1 The stress vector

It's time to examine **internal forces**. To examine the internal forces in an object, we make a cut along a plane. This plane has a certain **unit normal vector n**. The internal forces at a given position are now indicated by the **stress vector t(n)**. (Note that the stress vector can be seen as a surface force. That's why it is also denoted by $\mathbf{t}$.)

Let's suppose that we know the stress vector $\mathbf{t}$ at a given point for a given normal vector $\mathbf{n}$. We can then also find **normal component $\mathbf{t_n}$** (the stress normal to the cutting plane) and the **tangential component $\mathbf{t_s}$** (the stress parallel to the cutting plane). This can be done using

$$\mathbf{t_n} = (\mathbf{t} \cdot \mathbf{n})\mathbf{n} \qquad \text{and} \qquad \mathbf{t_s} = \mathbf{t} - \mathbf{t_n} = \mathbf{t} - (\mathbf{t} \cdot \mathbf{n})\mathbf{n}. \tag{2.1}$$

2.2 The stress tensor

There is, however, one small problem. The stress vector $\mathbf{t}$ depends on the on the cutting plane normal vector $\mathbf{n}$. To know the exact stress distribution, we need to know $\mathbf{t}$ for every $\mathbf{n}$. This may seem like a lot of work. Luckily, there is a trick (originating from the balance of momentum) called the stress tensor.

The **stress tensor** $[\sigma_{ij}]$ is a 3×3 matrix. It has thus 9 coefficients σ_{ij}. Once these parameters are known, the stress vector $\mathbf{t}$ for any unit normal vector $\mathbf{n}$ can be found using

$$\mathbf{t}(\mathbf{n}) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}. \tag{2.2}$$

The question now remains, how can we find the stress tensor? To do that, we have to first find the stress vector $\mathbf{t}$ for three (linearly independent) normal vectors $\mathbf{n}$. (It is often convenient to choose the three unit normal vectors $\mathbf{e_1}$, $\mathbf{e_2}$ and $\mathbf{e_3}$.) We should then find the corresponding stress vectors $\mathbf{t_1}$, $\mathbf{t_2}$ and $\mathbf{t_3}$. Inserting all these data into equation (2.2) gives us 9 equations and 9 unknowns. The unknown coefficients can then be solved.

When solving for the coefficients, you can use a small trick. You can use that the stress tensor is symmetric. (This can be derived from balance of angular momentum.) So we have

$$\sigma_{12} = \sigma_{21}, \qquad \sigma_{13} = \sigma_{31} \qquad \text{and} \qquad \sigma_{23} = \sigma_{32}. \tag{2.3}$$

2.3 Stress tensor eigenvalues and eigenvectors

The stress tensor $[\sigma_{ij}]$ has three eigenvalues $\sigma^{(1)}$, $\sigma^{(2)}$ and $\sigma^{(3)}$. These eigenvalues are called the **principal stresses**. Because the stress tensor is symmetric, these eigenvalues must be real. We usually order them such that $\sigma^{(1)} \geq \sigma^{(2)} \geq \sigma^{(3)}$.

Of course, there are eigenvectors $\mathbf{n^{(1)}}$, $\mathbf{n^{(2)}}$ and $\mathbf{n^{(3)}}$ corresponding to these eigenvalues. Usually, these eigenvectors are normalized, such that their length $|\mathbf{n}|$ is one. These vectors are called the **principal stress directions**. It can be shown that they are mutually perpendicular. Because of this, they together form an orthogonal basis, called the **principal stress basis**.

There is something special about this basis. Previously, we have built our stress tensor $[\sigma_{ij}]$ with respect to our normal Cartesian basis $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$. If we, however, build it with respect to the principal stress basis, we find a very peculiar stress tensor, being

$$[\sigma_{ij}] = \begin{bmatrix} \sigma^{(1)} & 0 & 0 \\ 0 & \sigma^{(2)} & 0 \\ 0 & 0 & \sigma^{(3)} \end{bmatrix}. \tag{2.4}$$

2.4 Relevance of principal stresses and their directions

You may wonder, what are these principal stresses and principal stress directions good for? Well, the principal stresses are used in many stress criterions. For example, there is the **tresca criterion**, demanding that

$$\max\left(|\sigma^{(1)} - \sigma^{(2)}|, |\sigma^{(1)} - \sigma^{(3)}|, |\sigma^{(2)} - \sigma^{(3)}|\right) \leq \sigma_y, \tag{2.5}$$

where the critical value σ_y is known as the **(initial) yield stress**. Similarly, there is the **Huber-von Mises-Hencky criterion**, demanding that

$$\sigma_m = \sqrt{\frac{\left(\sigma^{(1)} - \sigma^{(2)}\right)^2 + \left(\sigma^{(1)} - \sigma^{(3)}\right)^2 + \left(\sigma^{(2)} - \sigma^{(3)}\right)^2}{2}} \leq \sigma_y, \tag{2.6}$$

where σ_m is the **maximum distortion energy**.

The principal stress directions are also important. They are closely related to the directions and planes in which failure will initiate and propagate. This data is important when trying to optimize a structure.