Displacements and strains

After examining forces and stresses, we will now examine displacements and strains. How are they defined? And what can we do with them? We also examine their relation with stresses.

1 Definitions of the displacements and strains

1.1 Introduction of the strain tensor

Let's suppose we have some object Ω, we're deforming. Let's examine some point P. We call its initial position $\mathbf{x}$ and its final position $\mathbf{y}$. The **displacement** of P (its movement) then is

$$\mathbf{u} = \mathbf{y} - \mathbf{x}. \tag{1.1}$$

However, we are usually interested in the deformations of the material. The movement of some point P doesn't say much about that. To examine the deformations, we use **displacement gradients** $\partial u_i / \partial x_j$. In fact, the **strain tensor** is defined as

$$[\epsilon_{ij}] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}. \tag{1.2}$$

Note that the strain tensor is symmetric.

1.2 The meaning of the strain tensor

So, what is the use of the strain tensor? Well, it is closely related to the deformations in the object. To see how, we examine two points P and Q, originally being a very small distance l_0 apart. Their new relative distance is l. We call the unit vector in the direction of the line PQ the vector $\mathbf{n}$. The **relative elongation** ϵ of the distance PQ (also known as the normal strain) can then be approximated by

$$\epsilon(\mathbf{n}) = \frac{l - l_0}{l} \approx \sum_{i,j=1}^{3} \epsilon_{ij} n_i n_j = \mathbf{n}^T [\epsilon_{ij}] \mathbf{n}. \tag{1.3}$$

Using this, we can more closely examine the meaning of the strain tensor. If we examine the relative elongation in the direction of the Cartesian unit vector $\mathbf{e_1}$, then we find that $\epsilon(\mathbf{e_1}) = \epsilon_{11}$. Similarly $\epsilon(\mathbf{e_2}) = \epsilon_{22}$ and $\epsilon(\mathbf{e_3}) = \epsilon_{33}$. So the diagonal components simply indicate normal strain.

The next question is, what do the non-diagonal terms of the strain tensor mean? They indicate a change in angle of two lines that were previously perpendicular. (It's also known as the shear strain.) Let's examine two lines PQ and PR. PQ is in the direction of $\mathbf{e_1}$, while PR is in the direction of $\mathbf{e_2}$. Their relative angle is thus $\pi/2$. It can be shown that, after deformation, their relative angle is $\pi/2 - \epsilon_{12} - \epsilon_{21} = \pi/2 - 2\epsilon_{12}$.

1.3 Principal strains and their directions

Just like the stress tensor $[\sigma]$, also the strain tensor $[\epsilon]$ is symmetric. This means that its three eigenvalues $\epsilon^{(1)}$, $\epsilon^{(2)}$ and $\epsilon^{(3)}$, called the **principal strains**, are all real. The three corresponding eigenvectors $\mathbf{m^{(1)}}$, $\mathbf{m^{(2)}}$ and $\mathbf{m^{(3)}}$, called the **principal strain directions**, are mutually perpendicular. Together, they form the **principal strain basis**.

Previously, we have built our strain tensor $[\epsilon_{ij}]$ with respect to our normal Cartesian basis $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$. If we, however, build it with respect to the principal strain basis, we get the strain tensor

$$[\epsilon_{ij}] = \begin{bmatrix} \epsilon^{(1)} & 0 & 0 \\ 0 & \epsilon^{(2)} & 0 \\ 0 & 0 & \epsilon^{(3)} \end{bmatrix}. \tag{1.4}$$

All non-diagonal terms are zero. There is thus no shear strain. So we can conclude that, with respect to the principal strain basis, all perpendicular angles remain perpendicular.

You may wonder whether the principal stress directions and principal strain directions are the same. They usually are not. Only for **isotropic materials** (materials with the same properties in every direction) will these directions coincide.

1.4 Finding the displacement field from the strain tensor

Let's suppose we know the strain tensor $[\epsilon_{ij}]$ at every given position $\mathbf{x}$. Can we then find the displacements? Well, it turns out that we almost can do that. Only the so-called **rigid body modes**, being pure translation and rotation (without any deformation), can't be included. However, by using appropriate boundary conditions, we can get rid of these rigid body modes.

So how do we find the displacement field? Since this is a rather difficult process, we only consider the two-dimensional **plane strain** case. So $\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = 0$. The first step is to check whether the strains ϵ_{11}, ϵ_{12} and ϵ_{22} are **integrable**. To do this, we need to check the **compatibility equation**, being

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} = 0. \tag{1.5}$$

If this equation holds, there is a solution.

The next step is to use $\epsilon_{11} = \partial u_1/\partial x_1$ and $\epsilon_{22} = \partial u_2/\partial x_2$. In other words, we need to integrate ϵ_{11} and ϵ_{22} with respect to x_1 and x_2, respectively. This results in certain unknown functions. These functions can often be determined using $\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$ up to certain unknown constants. In the end, we should remain with a solution with a few unknown constants, indicating the rigid body modes.

1.5 The rotation tensor

Sometimes deformations aren't the only thing we're interested in. Rotations can also be important. To examine them, we use the **infinitesimal rotation tensor**, defined as

$$[\omega_{ij}] = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) & 0 \end{bmatrix}. \tag{1.6}$$

Note that this matrix is not symmetric. In fact, it is anti-symmetric ($\omega_{ij} = -\omega_{ji}$). The tensor above contains information about the **average rotation**. However, we'll not go into detail about this.

2 The constitutive relations

Constitutive relations relate loads with displacements. Or equivalently, they relate the stress tensor with the strain tensor. What are the relationships between those two tensors?

2.1 Linearly elastic solids

We want to find a relationship between the stress tensor and the strain tensor. There are, however, many types of materials. For some weird materials, the stress depends on the strain, the strain rate and the loading history.

Luckily, for most materials, the stress (approximately) only depends on the strain. And it does this in a linear way. Such materials are called **linear elastic solids**. For these materials, the stress tensor and the strain tensor can be related by a linear relation, such as

$$\sigma_{ij} = \sum_{k,l=1}^{3} C_{ijkl}\epsilon_{kl}. \tag{2.1}$$

The numbers C_{ijkl} are called the **elastic coefficients**. There are 81 of these components. Together, they form the **elasticity tensor** $[C_{ijkl}]$ (also denoted as $\mathbb{C}$). Usually, these components need to be determined experimentally. However, they are not all independent. So, we can apply some tricks.

2.2 Voigt's notation

We know that both the stress tensor and the strain tensor are symmetric. (So $\sigma_{ij} = \sigma_{ji}$ and $\epsilon_{ij} = \epsilon_{ji}$.) Because of this, we also must have $C_{ijkl} = C_{ijlk}$ and $C_{ijkl} = C_{jikl}$. (These relations are called **minor symmetries**.) So, instead of 81 independent coefficients, we now only have 36. And because of this, we can also write equation (2.1) as

$$\sigma_{ij} = C_{ij11}\epsilon_{11} + C_{ij22}\epsilon_{22} + C_{ij33}\epsilon_{33} + C_{ij23}(2\epsilon_{23}) + C_{ij13}(2\epsilon_{13}) + C_{ij12}(2\epsilon_{12}) \tag{2.2}$$

However, since there are only 36 independent coefficients, it's not useful to write 81 coefficients down every time. That's why **Voigt's notation** is often convenient. In Voigt's notation, the stress and strain tensors aren't written as 3×3 matrices, but as 6×1 vectors. Also, the elasticity tensor $\mathbb{C}$ is written as a 6×6 matrix. This gives us the following relation

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{bmatrix} = \begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ 2\epsilon_{23}\\ 2\epsilon_{13}\\ 2\epsilon_{12}\end{bmatrix}. \tag{2.3}$$

It's important to note the order of the coefficients. (In other books this order may be different.) Also note the twos in the strain vector.

The elasticity tensor above has another interesting property. By examining elastic strain energy, it can be shown that $C_{ijkl} = C_{klij}$. (This is a so-called **major symmetry**.) This implies that the elasticity tensor above is symmetric. So there are only 21 independent coefficients left.

2.3 The compliance matrix

If we know the strain and the elasticity tensor, then we can find the stress. But, we can also do it the other way around. For this, we use the **compliance tensor** $\mathbb{S} = \mathbb{C}^{-1}$. This then gives us that

$$\bar{\epsilon} = \mathbb{S}\sigma. \tag{2.4}$$

By the way, with $\bar{\epsilon}$ we mean the new strain vector (with the added twos).

2.4 Material symmetries

We still have 21 independent properties. But we can often reduce that number, due to **geometrical symmetries** in the material. Some materials have no such intrinsic symmetries. They are called **triclinic materials** and need to be described by 21 independent variables.

Some materials have a plane of symmetry. A material has such a **plane of summetry** if, after a reflection about that plane, is indistinguishable from the original material.

A material that has only one such plane of symmetry is called **monoclinic**. For such a material, eight of the coefficients will be zero. So there are 13 remaining independent coefficients. A material that has three mutually perpendicular planes of symmetry is called **orthotropic**. Such a material has 9 independent coefficients.

A material can also be **transversely isotropic**. In this case the material has three planes of symmetry, with an angle of 60° between them. (Like in a honeycomb structure.) In this case, there are only 5 independent coefficients.

2.5 Isotropic materials

For some materials every plane is a plane of symmetry. Such materials are called **isotropic materials**. Such materials have only two independent properties. We can write the elasticity tensor for these materials as

$$[C_{IJ}] = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}. \tag{2.5}$$

The parameters λ and μ are called the **Lamé coefficients**. μ is also known as the **shear modulus**. (λ has no clear physical meaning.) From the above matrix, we can also directly find that

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\delta_{ij}\sum_{k=1}^{3}\epsilon_{kk}. \tag{2.6}$$

By the way, the symbol δ_{ij} is the **Kronecker delta** and is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.7}$$

We can also invert the above relation to express the strain ε as a function of the stress σ. We then get

$$\epsilon_{ij} = \frac{1}{E}\left((1+\nu)\sigma_{ij} - \nu\delta_{ij}\sum_{k=1}^{3}\sigma_{kk}\right). \tag{2.8}$$

Here E is the **elastic modulus** and ν is **Poisson's ratio**.