Cell complexes

When examing fluids, it is often be wise to split the space up in cells. If we then know the relation between cells, we can perform calculations with them. This chapter examines the tricks we can apply with such cells. We also examine the relation with the actual integral equations.

1 Introduction

1.1 What is constitutive modelling?

Let's examine some mechanical system. The state of the system is described by **configuration variables**. These are variables like position, velocity and acceleration. This state is influenced by the so-called **source terms**. These are variables like force, stress and pressure.

Each of these two types of variables lives in its own realm. They can be equated with each other. By using integral relations and such. That's what we'll look at first in this summary.

However, equating configuration variables to source terms is more difficult. To equate variables from different realms, we need material parameters and other physical constants. The resulting relations form the **constitutive model**.

1.2 Mathematical operators

Many equations apply on different kinds of object. Some equations apply on volumes, others on planes and others on lines. Luckily, there are mathematical equations with which these equations can be related. For example, the **divergence theorem** relates a vector field $\mathbf{A}$ on a surface to a scalar field div $\mathbf{A}$ on a volume. It does this according to

$$\int_{\partial\Omega} \mathbf{A} \cdot d\mathbf{S} = \int_{\Omega} \text{div } \mathbf{A}\, dV. \tag{1.1}$$

So the **divergence operator** div relates the **space of surfaces** H_S to the **space of volumes** H_V. Similarly, according to **Stokes' theorem**, the **curl operator** curl relates the **space of lines** H_L to the space of surfaces H_L. Finally, the **gradient operator** grad relates the **space of points** H_P to the space of lines. This gives us the following mappings.

$$H_P \xrightarrow{\text{grad}} H_L \xrightarrow{\text{curl}} H_S \xrightarrow{\text{div}} H_V. \tag{1.2}$$

The above mapping is kind of special. When we map two operators in a row, we always get zero. (Mathematically speaking, we say that the **null space** of one operator coincides with the **range** of the other.) So we have

$$\text{curl grad } = 0 \qquad \text{and} \qquad \text{div curl } = 0. \tag{1.3}$$

For this reason, the above sequence is called an **exact sequence**.

2 Definitions

2.1 Cell complexes and dual complexes

Let's consider the n-dimensional space $\mathbb{R}^n$. We can divide this space in several p**-cells**, where p indicates the dimension of the cell. For example, a point is a 0-cell, a line is a 1-cell, a surface is a 2-cell, a volume is a 3-cell, and so on.

The collection of all the p-cells is called a **(primal) cell complex**. A cell complex is usually denoted by K. We also define α_p as t he number of p-cells in K.

There can also be a **dual cell complex** $\tilde{K}$ corresponding to a primal cell complex K. In this case, the p-cells of $\tilde{K}$ lie in the $(n-p)$-cells of K, and vice verse. For example, in a 2-dimensional space (a plane), the points (0-cells) of $\tilde{K}$ lie in the surfaces (2-cells) of K and vice verse. Also, the lines (1-cells) of $\tilde{K}$ cross the lines (1-cells) of K.

The p-cells in a cell complex K are usually numbered. This numbering can be done arbitrarily. However, the numbering of the p-cells in a dual cell complex $\tilde{K}$ is not arbitrary. We just saw that p-cells in $\tilde{K}$ correspond to $(n-p)$-cells in K. These correspondings cells are (by convention) given the same number.

2.2 Faces and cofaces

Let's examine a p-cell Q. (For example, a 2-cell, or a surface.) The **faces** of Q are the $(p-1)$-cells that form the boundary of Q. (So the faces of Q are the boundaries of Q.) The **cofaces** of Q are the $(p+1)$-cells that have Q as a face. (So the cofaces of Q are the objects which Q bounds.)

2.3 Chains and cochains

Let's examine a cell complex K. The collection of all α_p p-cells in K is called a **p-chain**. A collection of only some p-cells is called a **p-sub-chain**. The **boundary** δC of a p-sub-chain C consists of all $p-1$-cells which bound C. (More strictly speaking, it consists of all $p-1$-cells which have exactly one item of C as a coface.)

With each p-chain, we can associate a set of α_p numbers/vectors $(b_1, b_2, \ldots, b_{\alpha_p})$. The function that assigns the numbers/vectors to the p-cells is called a **p-cochain**. Two p-cochains $a^{(p)}$ and $b^{(p)}$ can be added up. To do this, you simply have to add up the individual elements. So

$$a^{(p)} + b^{(p)} = (a_1 + b_1, a_2 + b_2, \ldots, a_{\alpha_p} + b_{\alpha_p}). \tag{2.1}$$

We can also take the integral over a certain p-chain, or p-sub-chain. To do this, we have to add up all the elements in the corresponding co-chain. So we have

$$\int_p a^{(p)} = \sum a_i. \tag{2.2}$$

2.4 The incidence matrix and the coboundary operator

p-cells are usually also given an orientation. For 0-cells (points), this can be either inward or outward. For 1-cells (lines), this can be in one direction along the line, or in the opposite direction. For 2-cells (surfaces), this can be clockwise or counterclockwise. And so on.

These orientations are necessary to find the **incidence coefficients** $e_{ij}^{(p+1),(p)}$. To find them, we need to examine the i-th $(p+1)$-cell and the j-th p-cell. The coefficients are then defined as

$$e_{ij}^{(p+1),(p)} = \begin{cases} 0 & \text{if the } p\text{−cell is not a face of the } (p+1)\text{−cell,} \\ 1 & \text{if the cells have the same orientation,} \\ -1 & \text{if the cells have opposite orientation.} \end{cases} \tag{2.3}$$

We can find the incidence coefficients for all combinations of $(p+1)$-cells i and p-cells j. If we put all these coefficients in an $\alpha_{p+1} \times \alpha_p$ matrix, we have found the **incidence matrix** $E^{(p+1),(p)}$.

The incidence matrix allows us to generate $(p+1)$-cochains from p-cochains. For example, we can say that

$$b^{(p+1)} = E^{(p+1),(p)} a^{(p)} = \delta a^{(p)}, \tag{2.4}$$

where the δ, called the **coboundary operator**, is just another way of writing the above equation. We find that the incidence matrix and the coboundary operator have some interesting properties. We can, for example, multiply the matrices $E^{(p+2),(p+1)}$ and $E^{(p+1),p}$. It can then be shown that

$$E^{(p+2),(p+1)} E^{(p+1),(p)} a^{(p)} = E^{(p+2),(p)} a^{(p)} = 0^{(p+2)}, \qquad \text{or} \qquad \delta\delta a^{(p)} = 0^{(p+2)}. \tag{2.5}$$

So applying the coboundary operator twice always leads to the **null cochain** (the cochain filled with only zeroes). This actually makes sense. Because we're in fact finding the boundary of a boundary. Imagine some arbitrary surface. Now take its boundary. (It's a line.) This line doesn't have any end points. So the boundary of the boundary of the surface simply doesn't exist. It works the same for objects in other dimensions.

2.5 Coboundaries and cocycles

Let's take a look at the coboundary operator more closely. Suppose we apply it on a $(p-1)$-cochain $a^{(p-1)}$. The resulting p-cochain $b^p = \delta a^{(p-1)}$ is the so-called **coboundary** of $a^{(p-1)}$. In fact, we call any p-cochain b^p that is the coboundary of some $(p-1)$-cochain $a^{(p-1)}$ a **p-coboundary**.

Some p-cochains $b(p)$ have a zero coboundary. So $\delta b^{(p)} = 0^{(p+1)}$. Such cochains are called **cocycles**. It is interesting to note that any p-coboundaries b^p is a cocycle. (This is because $\delta b^p = \delta\delta a^{(p-1)} = 0^{(p+1)}$. We'll see in the next paragraph why this holds.) However, not any cocycle is a coboundary.

Finally, we say that two p-cochains are **cohomologous** if their difference is a coboundary. So $a^{(p)}$ and $b^{(p)}$ are cohomologous if there is a $(p-1)$-cochain $c^{(p-1)}$ for which $a^{(p)} - b^{(p)} = \delta c^{(p-1)}$.

3 Connections to the real world

3.1 Again an exact sequence

Previously, we have seen that we can use incidence matrices (or equivalently, the coboundary operator) to transform p-cochains to $(p+1)$-cochains. We can, for example, transform 1-cochains, connected to lines, to 2-cochains, which correspond to surface. So we can map $H_L \to H_S$. Let's visualize what other mappings we can do. We then find that

$$H_P \xrightarrow{\delta} H_L \xrightarrow{\delta} H_S \xrightarrow{\delta} H_V. \tag{3.1}$$

These mappings are exactly the same as the mappings we could do with grad, curl and div. So the coboundary operator seems to be a good replacement for all the integral theorems. By the way, since $\delta\delta = 0$, the above sequence again is an exact sequence.

3.2 The connection with integrals

Let's examine some cell complex. We can choose a p-sub-chain C in this path. (For example, a set of connected lines.) We associate with this p-sub-chain, the p-cochain $\delta a^{(p-1)}$, where $a^{(p-1)}$ is some $(p-1)-cochain$. We can then integrate over the p-sub-chain C. It can then be shown that

$$\int_C \delta a^{(p-1)} = \delta_{ij} = \int_{\delta C} a^{(p-1)}. \tag{3.2}$$

δ_{ij} is the so-called **Kronecker delta**. This important relation is called the **(generalized) Stokes' theorem**. But what does this mean? Well, we can integrate over the sub-chain of $\delta a^{(p-1)}$. But we can also integrate over the boundary of the sub-chain of $a^{(p-1)}$. And, according to the above theorem, both ways give exactly the same result.

The generalized Stokes' theorem is, as its name implies, a generalized version of many integral relations. We can, for example, recall the **gradient operator** grad. It satisfies

$$\int_C (\text{grad } p) \cdot d\mathbf{s} = p(C_{end}) - p(C_{begin}). \tag{3.3}$$

This integral relation says the same as the generalized Stokes' theorem for $p = 1$. We can examine δp along a one-dimensional curve. We can also simply examine p at the endpoints of the curve. It gives the same result.

Transforming other integral relations to integrals of sub-chains goes similar. For example, consider the conservation law

$$\frac{d}{dt}\int_\Omega \phi \, d\Omega + \int_{\delta\Omega} \mathbf{F} \, d\mathbf{S} = \int_\Omega q \, d\Omega. \tag{3.4}$$

If we use cochains, then the above equation turns into

$$\frac{d}{dt}\int_\Omega \phi^{(3)} + \int_{\delta\Omega} F^{(2)} = \int_\Omega q^{(3)} \qquad \Rightarrow \qquad \frac{d}{dt}\phi^{(3)} + \delta F^{(2)} = q^{(3)}. \tag{3.5}$$

Note that we have used the generalized Stokes' theorem on the middle part. After this, we were allowed to remove the integral. The final equation has no integrals anymore. Nor is there any connection to the volume Ω. So, we see that using cell complexes and cochains is a great way to get rid of integrals and geometry. Since there is no

3.3 The Laplace equation

Now let's consider the Laplace equation $\Delta\phi$, or equivalently, div grad ϕ. How can we transform this equation? We can't use $\delta\delta\phi$, since that would simply give zero. (You may wonder why the Laplacian does not give zero. This is because grad maps from points to lines, and div from surfaces to volumes.) So how do we tackle this equation?

The trick lies in the dual complex. We saw that (in 3D space), lines correspond to surfaces, and points correspond to volumes. So we can start with points. Then we use the grad operator to get lines. We then move to the dual complex to get surfaces. After this, we use the div operator to get volumes. Finally, we can move back to our original cell complex. We can display these mappings as

$$\begin{array}{ccccccc} H_P & \xrightarrow{\text{grad}} & H_L & \xrightarrow{\text{curl}} & H_S & \xrightarrow{\text{div}} & H_V \\ \updownarrow * & & \updownarrow * & & \updownarrow * & & \updownarrow * \\ H_{\tilde{V}} & \xleftarrow{\text{div}} & H_{\tilde{S}} & \xleftarrow{\text{curl}} & H_{\tilde{L}} & \xleftarrow{\text{grad}} & H_{\tilde{P}}. \end{array} \tag{3.6}$$

In this mapping, every square corresponds to the Laplace operator. So in fact, $\Delta\phi = *\delta * \delta\ \phi$. The so-called **Hodge operator** $*$ is used to transfer between the primal complex K and the dual complex $\tilde{K}$. A **constitutive model** is a model which maps between the primal complex and the dual complex. So we find that the Hodge operator $*$ is, in fact, a constitutive model.