Application of the consitutive models

We now know how stresses and strains relate to each other. It's time to find out how we can use this to solve problems. First we examine static problems. We then move on to dynamic problems.

1 Static problems

1.1 Conditions and equations

When solving problems, the stress field should obey certain conditions. First let's take a look at what conditions there are. From the first chapter of this summary, we can recall the balance of linear momentum. It stated that

$$\int_{\partial\Omega} \mathbf{t}\,dA + \int_{\Omega} \mathbf{b}\,dV = \int_{\Omega} \rho\mathbf{a}\,dV. \tag{1.1}$$

For static problems, the acceleration is zero. If we rewrite the above equation, and split it up in components, we can then find that

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0. \tag{1.2}$$

This is the **balance of linear momentum** for static problems. It is our first condition. There are also the so-called **compatibility conditions**. They demand that

$$\frac{\partial^2 \epsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \epsilon_{kl}}{\partial x_i \partial x_j} = \frac{\partial^2 \epsilon_{ik}}{\partial x_j \partial x_l} + \frac{\partial^2 \epsilon_{jl}}{\partial x_i \partial x_k}. \tag{1.3}$$

Note that there are 81 different compatibility conditions, for every combination of i, j, k and l. There are often also boundary conditions. Sometimes the displacement in some direction u_i is set. At other times, the boundary traction $\hat{t}_i$ is set. In this case, you can use the stress tensor to find a relation for $\hat{t}_i$.

So our task is to find a stress field which satisfies all the conditions. With that, we can then find the displacement field. For that, we use the constitutive relations

$$\sigma_{ij} = \sum_{k,l=1}^{3} C_{ijkl}\epsilon_{kl}. \tag{1.4}$$

and the strain-displacement relations

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.5}$$

There is, however, one small problem. An analytical solution only exists for a few simple problems. Therefore the above equations are often used in numerical methods. Nevertheless, we will examine some analytical solutions now.

1.2 Plane stress case

The first case we examine is the **plane stress case**. Stress occurs only in a plane. Therefore $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$. The stress also only depends on the position on the plane. Thus $\sigma_{ij} = \sigma_{ij}(x_1, x_2)$. (Note that in general $\varepsilon_{33} \neq 0$.) We also assume that the material is **isotropic** (it has the same properties in every direction) and **homogeneous** (the material has the same properties at every point in the structure). Also, there are no body forces. (Thus $b_1 = b_2 = 0$.)

To find the stress distribution, it is handy to use a so-called **Airy stress function** ψ. We define ψ such that

$$\sigma_{11} = \frac{\partial^2 \psi}{\partial x_2^2}, \qquad \sigma_2 = \frac{\partial^2 \psi}{\partial x_1^2} \qquad \text{and} \qquad \sigma_{12} = -\frac{\partial^2 \psi}{\partial x_1 \partial x_2}, \tag{1.6}$$

if such a function exists. This has several advantages. We can see that the balance of linear momentum is now automatically satisfied. But what about the 81 compatibility equations? Well, it turns out that there are only 6 independent compatibility equations. And of these 6, only 1 actually matters. (The others are not important or automatically satisfied.) This equation demands that

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2\frac{\partial \epsilon_{12}}{\partial x_1 \partial x_2}. \tag{1.7}$$

We can now use the relations between stress and strain (the constitutive relations). This turns the above equation into a single compatibility equation for the stress function, being

$$\frac{\partial^4 \psi}{\partial x_1^4} + 2\frac{\partial^4 \psi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \psi}{\partial x_2^4} = 0. \tag{1.8}$$

All we have to do is find a stress function ψ which satisfies this compatibility equation, and any given boundary conditions. Once we have done that, we have solved our problem.

1.3 Plane strain case

We now examine the **plain strain case**. Now we the strain occurs only in a plane. So $\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = 0$. (But not $\sigma_{33} = 0$.) Also, $\epsilon_{ij} = \epsilon_{ij}(x_1, x_2)$. We again assume that the material is isotropic and homogeneous, and that there are no body forces.

We define the stress function the same as in the plane stress case. So,

$$\sigma_{11} = \frac{\partial^2 \psi}{\partial x_2^2}, \qquad \sigma_2 = \frac{\partial^2 \psi}{\partial x_1^2} \qquad \text{and} \qquad \sigma_{12} = -\frac{\partial^2 \psi}{\partial x_1 \partial x_2}, \tag{1.9}$$

After examining compatibility equations, we find that the only remaining equation again is

$$\frac{\partial^4 \psi}{\partial x_1^4} + 2\frac{\partial^4 \psi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \psi}{\partial x_1^4} = 0. \tag{1.10}$$

So the plane stress and the plane strain case work quite the same.

1.4 Finding a stress function

So how do we find an appropriate stress function ψ? To do this, we simply assume a form for ψ. Usually, an exponential form would do well. Therefore we assume that

$$\psi = \sum_m \sum_n c_{mn} x_1^n x_2^m. \tag{1.11}$$

Terms with $m+n \leq 1$ will drop out of all the compatibility equations. We therefore don't consider them. Also, terms with $m+n \geq 5$ usually aren't necessary to get a good solution. This prevents us a bit from getting an incredibly huge polynomial.

After having a form for ψ, we insert it into the compatibility equation. This gives us some relations for the unknown coefficients c_{mn}. We also try to match ϕ with the boundary conditions. This gives us even more relations for the unknown coefficients. In the end, all the coefficients should be solved for.

It's not always possible to let ψ match exactly with the boundary conditions. In this case, **Saint Venant's principle** should often be used. This principle states that, relatively far away from the boundary, the introduced loads have spread out. This can be used to let ψ approximately match the boundary conditions.

2 Dynamic Problems

2.1 The wave equation

Let's examine the **half-space**. This is a space such that there is material for every $x_1 > 0$. The boundary of the half-space is thus simply the $\mathbf{e_2}, \mathbf{e_3}$ plane. We load this half-space on its boundary by a uniform time-varying compressive load $\hat{\mathbf{p}}(t)$ in $\mathbf{e_1}$-direction. Now let's ask ourselves, what happens?

Now let's examine linear momentum in the $\mathbf{e_1}$-direction. We assume that there are no body forces ($\mathbf{b} = \mathbf{0}$). We then see that

$$\frac{\partial \sigma_{11}}{\partial x_1} = \rho \frac{\partial^2 u_1}{\partial t^2}. \tag{2.1}$$

Due to symmetry, there is only displacement in $\mathbf{e_1}$-direction. So $\epsilon_{22} = \epsilon_{33} = \epsilon_{12} = \epsilon_{13} = \epsilon_{23} = 0$ and $\epsilon_{11} = \partial u_1 / \partial x_1$. We also have $\sigma_{11} = (\lambda + 2\mu)\partial u_1 / \partial x_1$. This turns the above equation into

$$c_p^2 \frac{\partial^2 u_1}{\partial x_1^2} = \frac{\partial^2 u_1}{\partial^2 t}, \qquad \text{where} \qquad c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}. \tag{2.2}$$

c_p is called the **longitudinal (pressure) wave speed**. We can now see that the above equation is the **wave equation**, known from partial differential equations. Of course, a PDE should have initial conditions and boundary conditions. The **initial conditions** are often assumed to be

$$u_1(x, 0) = 0 \qquad \text{and} \qquad \frac{\partial u_1}{\partial t}(x_1, 0) = 0. \tag{2.3}$$

There is only one **boundary condition**. It is set at $x_1 = 0$ and is given by

$$\sigma_{11}(0, t) = (\lambda + 2\mu) \frac{\partial u_1}{\partial x_1}(0, t) = -\hat{\mathbf{p}}(t). \tag{2.4}$$

2.2 The solution of the wave equation

It's time to solve the wave equation. The general solution of the wave equation is given by

$$u_1(x_1, t) = f\left(t - \frac{x_1}{c_p}\right) + g\left(t + \frac{x_1}{c_p}\right). \tag{2.5}$$

f and g are functions that need to be chosen such that the initial and boundary conditions are satisfied. f denotes a wave that travels in the positive $\mathbf{e_1}$-direction. Similarly, g denotes a wave that travels in the negative $\mathbf{e_1}$-direction.

If we also include the initial conditions and boundary conditions, we can derive relations for f and g. In fact, these two functions are given by

$$f(\eta) = \begin{cases} \frac{1}{\rho c_p} \int_0^\eta \hat{\mathbf{p}}(\tau)\, d\tau & \text{for } \eta \geq 0 \\ 0 & \text{for } \eta \leq 0 \end{cases} \qquad \text{and} \qquad g(\xi) = 0 \text{ for } \xi \geq 0. \tag{2.6}$$

Combining this with the general solution, we can find that

$$u_1(x_1, t) = \begin{cases} \frac{1}{\rho c_p} \int_0^{t - x_1/c_p} \hat{\mathbf{p}}(\tau)\, d\tau & \text{for } t \geq x_1/x_p, \\ 0 & \text{for } t \leq x_1/c_p. \end{cases} \tag{2.7}$$

So the displacement field is now known. The stress distribution can also be solved for. We then find that

$$\sigma_{11}(x_1, t) = \begin{cases} -\hat{\mathbf{p}}(t - x_1/c_p) & \text{for } t \geq x_1/x_p, \\ 0 & \text{for } t \leq x_1/c_p. \end{cases} \tag{2.8}$$

We can see something quite interesting from this equation. When a force is introduced into the half-space, it travels through the half-space with velocity c_p. That's interesting to know.

2.3 Multiple layers with different properties

What happens if we have two layers A and B, having different material properties? The two layers each have different wave velocities, being

$$c_p^A = \sqrt{\frac{\lambda_A + 2\mu_A}{\rho_A}} \qquad \text{and} \qquad c_p^B = \sqrt{\frac{\lambda_B + 2\mu_B}{\rho_B}}. \tag{2.9}$$

Let's suppayer A starts at $x_1 = 0$. It ends at $x_1 = d$, which is also where the other layer starts. For times $t < d/c_p^A$, layer B will not notice any of the waves coming from the applied load. However, for $t \geq d/c_p^A$, there will be an **incident pulse** f_i acting on layer B. At the boundary between these layers, part of this pulse will be reflected. This is the **reflected pulse** f_r. Another part will be transmitted into layer B. This is the **transmitted pulse** f_t. So, for $t \geq d/c_p^A$, we have

$$u_1(x_1,t) = f_i\left(t - \frac{x_1}{c_p^A}\right) + f_r\left(1 + \frac{x_1}{c_p^A}\right) \text{ for } x_1 < d \quad \text{and} \quad u_1(x_1,t) = f_t\left(t - \frac{x_1}{c_p^B}\right) \text{ for } x_1 > d. \tag{2.10}$$

The question remains, what are these functions f_r and f_t? To find them, we have to use conditions. We know that the displacement at the boundary must remain the same for both layers. Due to Newton's third law, also the stress must remain continuous. So the conditions at the boundary are

$$u_1(d^-,t) = u_1(d^+,t) \qquad \text{and} \qquad \sigma_{11}(d^-,t) = \sigma_{11}(d^+,t). \tag{2.11}$$

By using this, we can find that

$$f_r = \frac{\rho_B c_p^B - \rho_A c_p^A}{\rho_B c_p^B + \rho_A c_p^A} f_i \qquad \text{and} \qquad f_t = \frac{2\rho_A c_p^A}{\rho_B c_p^B + \rho_A c_p^A} f_i. \tag{2.12}$$

It is often convenient to define the **ratio of longitudinal acoustic impedances** γ_p as

$$\gamma_p = \frac{\rho_B c_p^B}{\rho_A c_p^A}. \tag{2.13}$$

In this case, the above equations turn into

$$f_r = \frac{1-\gamma_p}{1+\gamma_p} f_i \qquad \text{and} \qquad f_t = \frac{2}{1+\gamma_p} f_i. \tag{2.14}$$

We can find similar relations for the stress propagation. These relations are

$$\sigma_{11}^{(r)} = \frac{\gamma_p - 1}{\gamma_p + 1}\sigma_{11}^{(i)} \qquad \text{and} \qquad \sigma_{11}^{(t)} = \frac{2\gamma_p}{\gamma_p + 1}\sigma_{11}^{(i)}. \tag{2.15}$$

2.4 Shear stress propagation

We have seen how normal stress propagates in a half-space. But what about shear stress? Let's assume a shear load $\hat{\mathbf{s}}(t)$ is applied on the half-space boundary, in the $\mathbf{e_2}$-direction. What happens?

This time we have $\epsilon_{12} = \frac{1}{2}\frac{\partial u_2}{\partial x_1}$, while $\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = \epsilon_{13} = \epsilon_{23} = 0$. We also have $\sigma_{12} = 2\mu\epsilon_{12} = \mu\frac{\partial u_2}{\partial x_1}$. This time, we can derive from balance of linear momentum that

$$c_s^2 \frac{\partial^2 u_2}{\partial x_1^2} = \frac{\partial^2 u_2}{\partial t^2}, \qquad \text{where} \qquad c_s = \sqrt{\frac{\mu}{\rho}}. \tag{2.16}$$

The quantity c_s is called the **transverse (shear) wave speed**. We again have a wave equation. The **initial conditions** now are

$$u_2(x,0) = 0 \qquad \text{and} \qquad \frac{\partial u_2}{\partial t}(x_1,0) = 0. \tag{2.17}$$

The boundary condition is again set at $x_1 = 0$. It is now given by

$$\sigma_{12}(0,t) = \mu \frac{\partial u_2}{\partial x_1}(0,t) = \hat{\mathbf{s}}(t). \tag{2.18}$$

We can solve the wave equation for u_2. We then find that

$$u_2(x_1,t) = \begin{cases} \frac{1}{\rho c_s} \int_0^{t-x_1/c_s} \hat{\mathbf{s}}(\tau)\, d\tau & \text{for } t \geq x_1/x_s, \\ 0 & \text{for } t \leq x_1/c_s. \end{cases} \tag{2.19}$$

Similarly, we can find a relation for the shear stress distribution. We now find that

$$\sigma_{12}(x_1,t) = \begin{cases} \hat{\mathbf{s}}(t - x_1/c_s) & \text{for } t \geq x_1/x_s, \\ 0 & \text{for } t \leq x_1/c_s. \end{cases} \tag{2.20}$$