Application of the conservation laws

It is finally time to actually put theory into practice. We have three conservation laws. Let's apply them!

1 Two parallel plates

1.1 The problem statement

Let's suppose we have two horizontal plates. One is positioned at $y = 0$ and the other at $y = D$. The top plate moves with a velocity u_e to the right. This causes a flow to be present between the plates. There is also a pressure distribution p between the plates, which also causes a flow.

We now make a few assumptions about the flow. We assume that the flow has a **constant density** $\rho = \rho_\infty$, that it is **steady** $\partial/\partial t$, that the flow is parallel to the plates ($v = w = 0$), that it doesn't vary in the z-direction ($\partial/\partial z$), and that there are no body forces $\mathbf{f_b}$.

1.2 Deriving the solution

Now let's solve the problem. Since ρ is constant, we can simplify the continuity equation to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{1.1}$$

Since $v = w = 0$, two terms cancel. We remain with $\partial u/\partial x = 0$. That's our first sub-result.

Now let's examine the momentum equation. If we write down the equation for all three components, we can see that a lot of terms cancel. We remain with

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}, \qquad \frac{\partial p}{\partial y} = 0 \quad \text{and} \quad \frac{\partial p}{\partial z} = 0. \tag{1.2}$$

The two rightmost equations imply that $p = p(x)$. If we look at the leftmost equation, we see that it is separated. Both sides must thus equal a constant c. This implies that

$$u(y) = \frac{c}{2\mu} y^2 + ay + b. \tag{1.3}$$

By using boundary conditions $u(0) = 0$ and $u(D) = u_e$ (the so-called **no-slip conditions**), we find that

$$u(y) = \frac{1}{2\mu} \left(\frac{dp}{dx} \right) \left(y^2 - Dy \right) + u_e \frac{y}{D}. \tag{1.4}$$

We have found the velocity distribution. We see that it consists of two parts. The left part originates from pressure differences, while the right part is caused by the moving plates. If there is no pressure difference ($dp/dx = 0$), then we have a so-called **planar Couette flow**. If, on the other hand, the plates are not moving ($u_e = 0$, but $dp/dx \neq 0$), then we have a **planar Poiseuille flow**.

2 Analysis the planar Poiseuille flow

2.1 Analyzing the velocity

Let's examine the planar Poiseuille flow now. So we assume the plates are not moving. We can now derive a lot of things from the solution. First, we can see that the **maximum velocity** u_{max} occurs at

$y = D/2$, exactly between the plates. Its magnitude is

$$u_{max} = -\frac{D^2}{8\mu}\left(\frac{dp}{dx}\right). \tag{2.1}$$

The minus sign makes sense. The velocity flows in the direction of a negative pressure gradient.

Now let's examine the **volume flow** $\dot{Q}$ (also known as the **volumetric flow rate**) flowing through the channel. It is given by

$$\dot{Q} = \int_0^D u(y)\,dy = -\frac{1}{12\mu}\left(\frac{dp}{dx}\right)D^3. \tag{2.2}$$

From this, we can derive the **mean velocity** $\bar{u}$. It is given by

$$\bar{u} = \frac{Q}{D} = -\frac{D^2}{12\mu}\left(\frac{dp}{dx}\right) = \frac{2}{3}u_{max}. \tag{2.3}$$

2.2 Analyzing the forces

The **wall shear stress** τ_w is given by

$$\tau_w = \mu\left(\frac{du}{dy}\right). \tag{2.4}$$

If we evaluate this at the walls ($y = 0$ and $y = D$), we find that this flow gives us

$$\tau_w = \frac{1}{2}D\left(\frac{dp}{dx}\right). \tag{2.5}$$

So the wall stress is independent of the viscosity μ. From this wall stress, we can derive the **friction coefficient** c_f. It is defined as

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho\bar{u}^2}. \tag{2.6}$$

If we apply this definition to our flow, we can find that

$$c_f = 12\frac{\mu}{\rho\bar{u}D} = \frac{12}{Re_D}. \tag{2.7}$$

The variable Re_D is the **Reynolds number**, with length D as the reference length.

3 Analysis of the planar Couette flow

3.1 Velocity and stress

Now let's examine the Couette flow. So we assume that $\partial p/\partial x = 0$. However, the top plate does move with a velocity u_e. The velocity distribution is thus given by

$$u(y) = u_e\frac{y}{D}. \tag{3.1}$$

The shear stress in the flow can be found using

$$\tau_{xy} = \mu\left(\frac{du}{dy}\right) = \mu\frac{u_e}{D}. \tag{3.2}$$

This shear stress is the whole reason why the flow is moving. It is caused by the moving top plate, and is, through the fluid, transferred to the bottom plate.

3.2 Finding the temperature distribution

It is time to apply the energy equation. After removing the zero terms, we find that

$$\frac{\partial}{\partial y}\left(\kappa \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(\mu u \frac{\partial u}{\partial y}\right) = 0. \tag{3.3}$$

We now have to assume that κ and μ are constant. (We say that $\kappa = \kappa_\infty$ and $\mu = \mu_\infty$.) So we can pull them out of the derivative. We also introduce the **enthalpy** $h = c_p T$, with c_p the **specific heat**. This turns the energy equation into

$$\frac{1}{P_r}\frac{\partial^2 h}{\partial y^2} + \frac{\partial}{\partial y}\left(u \frac{\partial u}{\partial y}\right) = 0, \qquad \text{where} \qquad P_r = \frac{\mu_\infty c_p}{\kappa_\infty}. \tag{3.4}$$

The number P_r is known as the **Prandtl number**. Integrating twice will give

$$h + \frac{1}{2} P_r u^2 = ay + b, \tag{3.5}$$

where a and b are constants. To find them, we use the boundary conditions $u(0) = 0$, $h(0) = h_w$ (the enthalpy at the bottom plate), $u(D) = u_e$ and $u(D) = h_e$ (the enthalpy at the top plate). This then gives us an expression for the enthalpy, being

$$h(y) = h_w + (h_e - h_w)\frac{y}{D} + \frac{1}{2} P_r u_e^2 \left(\frac{y}{D} - \left(\frac{y}{D}\right)^2\right). \tag{3.6}$$

And with this, the temperature distribution has been found. Something interesting can be seen here. If $h_e = h_w$, then the enthalpy isn't just constant. It is still parabolically distributed. This is caused by heat creation due to viscous effects.

3.3 Analysis of the heat flow

But we can do even more with the temperature (or enthalpy) distribution. From it, we can derive the **heat flow** $\dot{q}$, using

$$\dot{q} = -\kappa \frac{dT}{dy} = -\frac{\kappa}{c_p}\frac{dh}{dy} = -\frac{\mu}{D}\frac{h_e - h_w}{P_r} + \frac{1}{2}\frac{\mu}{D} u_e^2 \left(2\frac{y}{D} - 1\right). \tag{3.7}$$

Again, we see that, if $h_e = h_w$, there is still heat flow. It flows away from the center and goes to the plates, where it is dissipated. In fact, we can even find how much heat is dissipated at the walls. For that, we examine $\dot{q}(0)$ and $\dot{q}(D)$. We find them to be

$$\dot{q}(0) = -\frac{\mu}{D}\frac{h_e - h_w}{P_r} - \frac{1}{2}\frac{\mu}{D} u_e^2 \qquad \text{and} \qquad \dot{q}(D) = -\frac{\mu}{D}\frac{h_e - h_w}{P_r} + \frac{1}{2}\frac{\mu}{D} u_e^2. \tag{3.8}$$

In other words, the heat flow through the walls due to the viscous effects is $\frac{1}{2}\frac{\mu}{D} u_e^2$. This means that the total heat creation per second due to viscous effects is equal to $\frac{\mu}{D} u_e^2$ (since there are two walls). And that is interesting to know.