Trusses

1 Determinancy in Truss Structures

1.1 Introduction to determinacy

A **truss structure** is a structure consisting of members, connected by joints. Truss members are only subject to tension/compression.

Suppose we have a 2-dimensional truss structure. n is the number of **joints** (nodes) in the structure, m the number of **members** and r is the number of **reaction forces** acting on the structure. (So for a clamped beam $r = 3$, for a hinge support $r = 2$ and for a hinge on a roller support $r = 1$.) Whether the structure is **kinematically determinate** and **statically determinate** can be derived from figure 1.

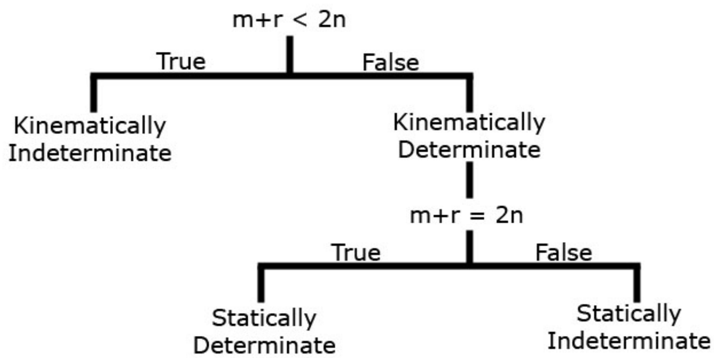

Figure 1: Overview of kinematical/statical determinacy

1.2 Kinematical determinacy

But what does it mean? A structure that is kinematically indeterminate can move (or parts of it can move), while kinematically determinate structures can not move at all. Statical determinacy is a bit more difficult to explain.

1.3 Statical determinacy

A statically determinate structure has just the right amount of members to make the structure also kinematically determinate. Remove 1 member, and the structure becomes kinematically indeterminate. Add 1 member, and the structure becomes statically indeterminate. The result of this is the following.

Let's suppose we have a statically determinate structure with a given applied load and truss geometry. The stress in any beam depends only on the cross-sectional area of that beam: $\sigma_i = f(A_i)$. Now suppose we have a statically indeterminate structure. The stress depends on the cross-sectional areas of all the beams: $\sigma_i = f(A_1 + \ldots + A_m)$. Also, to find the stresses, additional boundary conditions are required.

2 Stress Analysis in Truss Structures

2.1 Statically determinate trusses

Suppose at a later time we will receive data about a truss structure and want to calculate the forces in each member. To calculate this, we could write a computer program. But how can we let a program calculate the forces in every member?

For statically determinate structures, we can take the sum of forces in every node, both in x-direction and y-direction. We then get $2n$ equations. But what are our unknowns? The m internal forces in the members are unknown, and the r reaction forces are unknown as well. But since $m + r = 2n$ (the structure is statically determinate), we have the same amount of unknowns as equations. It can therefore be solved by a computer program.

When all the forces are known, it is easy to calculate the stresses present in the structure. To calculate the stress in beam i, just use

$$\sigma_i = \frac{F_i}{A_i}, \tag{2.1}$$

where F_i is the normal force in that beam and A_i is the cross-sectional area.

2.2 Statically indeterminate trusses

If the structure is statically indeterminate, then $m + r > 2n$, meaning that there are more unknowns then equations. Therefore additional boundary conditions are needed to calculate the stresses. These boundary conditions usually involve displacements.

Displacements are generally difficult to calculate. They don't only depend on the truss configuration. They also depend on the cross-sectional areas of the members. But there is a method with which this can be done, called the dummy load method. Let's take a look at that method now.

3 Dummy Load Method Derivation

3.1 Displacements and energy

A force on a structure always causes **displacements**. Energy is stored in such displacements. If P is the applied force on a structure and δ is the displacement, then the work done in the structure (and thus the energy stored in the structure) is

$$U = \int P \, d\delta. \tag{3.1}$$

In an **elastic (conservative) structure** the energy can be recovered completely. In a **plastic (non-conservative) structure** part of the energy is lost, causing permanent deformations.

It is usually assumed that δ is a linear combination of P, meaning that

$$P = k\delta, \tag{3.2}$$

where k is the **stiffness**. For truss members with uniform cross-section $k = \frac{EA}{L}$. This implies that

$$\frac{dU}{d\delta} = P, \qquad \frac{dU}{dP} = \delta. \tag{3.3}$$

This relation is called **Castigliano's theorem**. So for one truss member, the internal energy is given by

$$U = \int_0^l P \, d\delta = \int_0^l \frac{P}{k} dP = \frac{P^2}{2k} = \frac{P^2 L}{2AE}. \tag{3.4}$$

To derive the dummy load method for other load cases than just tension/compression, we can use table 3.1.

Type	Displacement	Energy
Tension/Compression Bar	$d\delta = \frac{F\,dx}{AE}$	$dU = \frac{1}{2}F\,d\delta = \frac{F^2\,dx}{2AE}$
Torsion Bar	$d\phi = \frac{T\,dx}{GJ}$	$dU = \frac{1}{2}T\,d\phi = \frac{T^2\,dx}{2GJ}$
Bending Beam	$d\theta = \frac{M\,dx}{EI}$	$dU = \frac{1}{2}M\,d\theta = \frac{M^2\,dx}{2EI}$
Shear Beam	$\gamma = \frac{VQ}{ItG} \approx \frac{V}{AG}$	$dU = \frac{1}{2}V\gamma dx = \frac{V^2Q\,dx}{2ItG} \approx \frac{V^2dx}{2AG}$

Table 1: Basic Structural Deformations

3.2 Multiple members and loads

Now look at a complete structure with n members with internal forces $F_1, \dots F_n$, caused by m applied loads $P_1, \dots, P_m$. The total energy stored in the system is

$$U = \sum_{i=1}^{n} \frac{F_i^2 L_i}{2A_iE_i}. \tag{3.5}$$

Suppose P_j is a virtual (nonexisting) force acting on some point j in the structure. The displacement of j in the direction of P_j now is

$$\delta_j = \frac{\partial U}{\partial P_j} = \sum_{i=1}^{n} \frac{\partial\left(\frac{F_i^2 L_i}{2A_iE_i}\right)}{\partial P_j} = \sum_{i=1}^{n} \frac{\frac{\partial\left(F_i^2\right)}{\partial P_j} L_i}{2A_iE_i} = \sum_{i=1}^{n} \frac{2F_i\frac{\partial F_i}{\partial P_j} L_i}{2A_iE_i} = \sum_{i=1}^{n} \frac{F_i f_i L_i}{A_iE_i}, \tag{3.6}$$

where f_i is defined as $f_i = \frac{\partial F_i}{\partial P_j}$. Let's take a closer look at this f_i. What is it? In fact, there is a linear relation between F_i and P_j. So we can say that $F_i = P_j f_i$. To find f_i, simply set $P_j = 1$ and calculate F_i.

3.3 Dummy load method

The **dummy load method** is a method to calculate displacements. It uses the relation that was just found (equation 3.6).

It can be used for statically determinate structures to calculate the displacements. This is the subject of the next part. It can also be used for statically indeterminate structures. In a statically indeterminate structure, displacements are necessary to calculate the forces in the structure.

4 Displacements in Statically Determinate Trusses

4.1 Step 1 - Calculate all the internal forces

Suppose we have a statically determinate structure and want to find the displacement of some point j. To use the dummy load method, we have to calculate the internal forces $F_1, \dots, F_n$ first.

4.2 Step 2 - Calculate the force derivatives

Now we know F_i for every i. Of course we also know the shape of the structure, so we know L_i, A_i and E_i for every i. The only unknowns are $f_1, \dots, f_n$. Use the following steps to find them.

- Remove all external forces $P_1, \dots P_m$ from the structure.
- Place a load $P_j = 1$ at point j, in the direction of which you want to know the displacement. Note that if you want to find the actual displacement vector of a point, you need to perform this method twice. Once for each direction.
- Calculate the force in all the members, due to this load P_j. The value of f_i is now the internal force of member i that results from this calculation.

4.3 Step 3 - Calculate the displacement

Now F_i and f_i are known for every i. To calculate the displacement in the specified direction, you must use

$$\delta_j = \sum_{i=1}^{n} \frac{F_i f_i L_i}{A_i E_i}. \tag{4.1}$$

With this method the displacement of every node in the structure can be calculated.

5 Statically Indeterminate Trusses

5.1 Superposition

In the last chapter we saw that a structure with m members, n nodes and r reaction forces is statically indeterminate if $m + r > 2n$. The **degree of statical indeterminacy** is defined as $d = m + r - 2n$. This is also the amount of members you can remove until the structure becomes statically determinate.

The dummy load method for indeterminate structures uses the principle of **superposition**. This states that for multiple loads, the displacements caused by the individual loads can be added up. So if there is a structure with multiple loads acting on it, you can calculate the displacement caused by every load individually, and eventually add them all up.

5.2 Step 1 - Remove members until statical determinacy is reached

But how can we use this on a statically indeterminate structure? To keep things simple, we assume that the degree of statical indeterminacy is 1. Let's suppose that member j is between nodes A and B. Also suppose that if we remove member j, the structure becomes statically determinate.

5.3 Step 2 - Calculate displacement due to external forces

We now have a statically determinate structure. Therefore, we can calculate the force in every member $F_1^{ext}, \dots F_m^{ext}$ (except member j, of course) due to the externally applied loads. Since member j is removed, it can't carry any loads. So we assume that $F_j^{ext} = 0$. Keep in mind that these are not the loads that are actually present in the structure, since we removed a member!

To use the dummy load method, we want to calculate the change of the distance AB. So we assume that there is a unit load (of size $1N$) acting on both A and B, pointing inward (where member j was). The forces in every member $f_1, \dots, f_m$ (due to this unit load) can now be calculated. The shortening of distance AB in the determinate structure, due to the external forces, is

$$\delta_{AB}^{ext} = \sum_{i=1}^{m} \frac{F_i^{ext} f_i L_i}{A_i E_i}. \tag{5.1}$$

In the above summation, we have $F_j^{ext} = 0$. So the entire j-term will vanish.

5.4 Step 3 - Calculate displacement due to the removed member

We have assumed member j wasn't present. But of course it is present. The member is actually causing a force F_j on the structure at points A and B. And to calculate the displacement due to member j, we need to take into account this force.

So we assume there is a force F_j acting on both A and B, pointing inward (where member j was). The forces in all members due to this internal load, $F_1^{int}, \ldots, F_m^{int}$, can now be calculated. In fact, we can take a short cut using

$$F_i^{int} = F_j f_i, \tag{5.2}$$

for every member $i \neq j$. Note that F_j^{int} isn't really defined. But if we define $F_j^{int} = F_j$ and $f_j = 1$, then equation 5.2 also holds for $i = j$.

Using equation 5.2 and $f_j = 1$, we can derive that the change of distance AB, due to the force caused by member j, is

$$\delta_{AB}^{int} = \sum_{i=1}^{m} \frac{F_i^{int} f_i L_i}{A_i E_i} = F_j \sum_{i=1}^{m} \frac{f_i^2 L_i}{A_i E_i}. \tag{5.3}$$

5.5 Step 4 - Equate displacements to zero

We now know the change of distance AB due to external forces (being δ_{AB}^{ext}) and the change of distance AB due to internal forces (being δ_{AB}^{int}). To find F_j, we can use the simple relation

$$\delta_{AB}^{ext} + \delta_{AB}^{int} = 0. \tag{5.4}$$

The only unknown in this relation is F_j, so it can be solved. By the way, we will not show why this relation is true. (The explanation is too long for a summary.)

Now we finally know F_j. We still don't know the other forces in the structure. To find the actual forces, we just have to sum up the part caused by external loads and the part caused by internal forces. So the actual forces in every member can be calculated using

$$F_i^{act} = F_i^{ext} + F_i^{int} = F_i^{ext} + F_j f_i. \tag{5.5}$$

6 Actuation

6.1 Actuated members

Truss members can be actuated by external effects. One of the most common effects is a change in temperature. A material subject to a change in temperature is assumed to elongate/shorten according to

$$\varepsilon_T = \alpha \, \Delta T, \tag{6.1}$$

where α is the **coefficient of thermal expansion** (CTE). Other ways of actuation can be treated similarly, so we will only examine actuation by temperature. The total strain now is

$$\varepsilon = \varepsilon_T + \varepsilon_M = \alpha \, \Delta T + \frac{P}{AE}, \tag{6.2}$$

where ε_M is the strain due to mechanical forces.

6.2 Actuation effects

But what effect does this have on the dummy load method? In equation 5.1 we saw the part F_i^{ext}/A_iE_i. This is equal to the mechanical strain ε_M. We should replace this by the new strain. So instead of using equation 5.1 we use

$$\delta_{AB}^{ext} = \sum_{i=1}^{m} f_i \varepsilon_i L_i = \sum_{i=1}^{m} f_i L_i \left(\alpha \, \Delta T + \frac{F_i^{ext}}{A_i E_i} \right). \tag{6.3}$$