Plates

1 Basic Stress Analysis in Plates

1.1 Uniaxial stress state

Suppose we have a plate with thickness t, width w (in the x-direction) and length L (in the y-direction). Also suppose that the plate is rigidly connected at the bottom and loaded by a force P at the top, such that the stresses are uniformly divided over the plate. Now we have a plate with a **uniaxial stress state**. The stresses are now given by

$$\sigma_y = \frac{P}{wt}, \qquad \sigma_x = 0. \tag{1.1}$$

From these stresses we can derive the strains. If ν is **Poisson's ratio** of the material, then

$$\varepsilon_y = \frac{\sigma_y}{E} = \frac{P}{Ewt}, \qquad \varepsilon_x = -\nu\varepsilon_y = -\frac{\nu P}{Ewt}. \tag{1.2}$$

1.2 Biaxial stress state

What if we constrain the plate of the last paragraph on the left and the right side? We then actually set ε_x to be zero. This causes the stresses in the plate to change. Let's call the strain in x-direction ε_x and the strain in y-direction ε_y. We can now solve this problem, using the basic equations

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E} = \frac{P}{Ewt} - \nu\frac{R_x}{ELt}, \qquad \varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} = \frac{R_x}{ELt} - \nu\frac{P}{Ewt}, \tag{1.3}$$

where R_x is the reaction force in horizontal direction. Since $\varepsilon_x = 0$, we can derive that

$$R_x = \nu\frac{PL}{w}. \tag{1.4}$$

Using this, the stresses can be found. They are

$$\sigma_y = \frac{P}{wt}, \qquad \sigma_x = \frac{R_x}{Lt} = \nu\frac{P}{wt} = \nu\sigma_y. \tag{1.5}$$

1.3 Multiple materials

When a plate consists of individual parts of different materials, the situation is more complicated. Still the equations of the previous paragraphs can be applied, but several other compatibility equations are necessary. There are no basic equations for that, but there are a few tricks that often need to be used.

- The stresses in the individual plate parts, in both x and y-direction can be expressed as a force using $F = \sigma A$, where A is the cross-sectional area. Using "sum of the forces is zero" for individual plate parts can give several compatibility equations.
- Often a plate is constrained in horizontal direction. In that case you can use the rule: "The sum of the horizontal displacements is zero."
- Finally, a plate is often constrained at the bottom and stressed uniformly at the top. In that case you can use the rule: "The vertical displacement of every part is equal."

2 Stress Analysis in Plates Using Mohr's Circle

2.1 Present stresses

In the previous chapter, we have only considered uniform load cases on rectangular uniform plates. Such simple geometries are often not present. For plates, stress analysis is therefore usually too complex. In that case stresses can not be calculated directly. Nevertheless, the stresses are present.

In a 2-dimensional plate, three kinds of stresses are present, being σ_x (normal stress in x-direction), σ_y (normal stress in y-direction) and τ_{xy} (shear stress in the xy-plane). In 3-dimensional plates, six kinds of stresses are present, being σ_x, σ_y, σ_z, τ_{xy}, τ_{xz} and τ_{yz}. However, we only consider 2-dimensional plates from now on.

2.2 Mohr's circle

The stresses in a plate can not be calculated, but they can be measured. Suppose we measure σ_x, σ_y and τ_{xy} at a certain point. We get certain values. Suppose we now change (rotate) our coordinate system and measure σ_x, σ_y and τ_{xy} at the same point again. We now get different values. If we keep doing this for different (rotated) coordinate systems, and plot all the data we find in a $\sigma - \tau$ coordinate system, we get a circle, as can be seen in figure 1. This circle is called **Mohr's circle**.

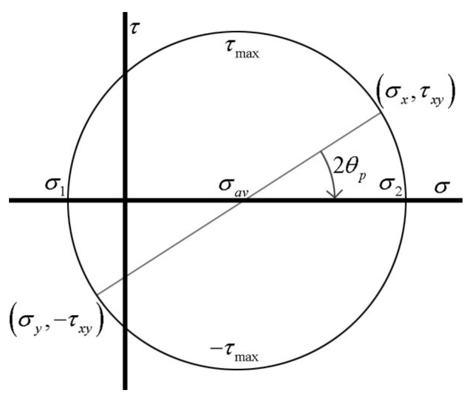

Figure 1: Mohr's Circle

2.3 Circle properties

Now let's take another look at figure 1. Suppose we have done a measurement and gotten the values σ_x, σ_y and τ_{xy}. We mark them in a graph and then draw a line between them. Where the line crosses the σ-axis is the **average stress** σ_{av}, which can also be calculated using

$$\sigma_{av} = \frac{\sigma_x + \sigma_y}{2}. \tag{2.1}$$

Also the **radius** of the circle can be calculated, using

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}. \tag{2.2}$$

2.4 Maximum stress

If we take measurements for different coordinate axes, the stresses will be different. What we are interested in, are the maximum stresses. The **maximum shear stress** is

$$\tau_{max} = R. \tag{2.3}$$

To find the **minimum and maximum normal stresses**, we can use

$$\sigma_1 = \sigma_{av} - R, \qquad \sigma_2 = \sigma_{av} + R. \tag{2.4}$$

Note that σ_1 and σ_2 can be both positive (in case of tension in both directions), both negative (in case of compression in both directions), or it is possible that $\sigma_2 > 0$ and $\sigma_1 < 0$. For figure 1 this means that the circle can move to the left and to the right for different load cases. However, the circle can not move upward or downward - the circle center is always at the σ-axis.

2.5 Stress directions

It is often handy to know in which direction maximum stresses occur. This can also be derived from Mohr's circle. Let's start rotating our coordinate system and do measurements. At some moment, when we have rotated the coordinate system by an angle θ_p, we measure maximum normal stress. In figure 1 this angle θ_p is visualized. From this figure we can see that

$$\tan 2\theta_p = \frac{\tau_{xy}}{\frac{\sigma_x - \sigma_y}{2}} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}. \tag{2.5}$$

Let's define θ_s to be the angle at which maximum shear stress occurs. We can now see that

$$\theta_s = \theta_p \pm 45°. \tag{2.6}$$