Beams

1 Stress Analysis in Statically Determinate Beams

1.1 Beam rules

In a truss structure only **normal forces** are present in the members. In a beam, also **shear forces** and **bending moments** can be present. This has effects on the deformation of those beams. Two general rules apply for that.

- Plane sections normal to the longitudinal axis remain normal after deformation.
- The thickness of the beam is unchanged.

When a beam has three support reactions acting on it, the beam is **statically determinate**. The reaction forces can then be solved in an easy way. If there are, however, more reaction forces acting on it, the beam is called **statically indeterminate**.

1.2 Normal stress

To calculate the **normal stress** σ at a certain point in the beam, we can use

$$\sigma = \sigma_N + \sigma_M = \frac{F}{A} - \frac{My}{I}, \tag{1.1}$$

where M is the bending moment at the specified point, I is the second area moment of inertia and y is the vertical distance from the COG of the cross-section. Note that $\sigma_N = \frac{F}{A}$ is the part due to normal forces and $\sigma_M = \frac{My}{I}$ is the part due to bending moments. The minus sign is present due to sign convention.

1.3 Shear stress

To calculate the **shear stress** at a certain point, we can use

$$\tau = \frac{VQ}{It}, \tag{1.2}$$

where V is the shear force present, Q is the first area moment of inertia, I is the second area moment of inertia and t is the thickness at the part where the shear stress is calculated.

1.4 Rotations and displacements

The rotations and displacements of a beam can be calculated using the so-called forget-me-nots. If a beam of length L, E-modulus E and moment of inertia I is subject to either a bending moment M, a load P or a distributed load q, then the rotation θ and the displacement δ can be found by using

$$\theta = \frac{ML}{EI}, \qquad \theta = \frac{PL^2}{2EI}, \qquad \theta = \frac{qL^3}{6EI}, \tag{1.3}$$

$$\delta = \frac{ML^2}{2EI}, \qquad \delta = \frac{PL^3}{3EI}, \qquad \delta = \frac{qL^4}{8EI}. \tag{1.4}$$

For complicated structures, applying these equations isn't always very easy. So there exists another method to find the rotation and displacement of a beam.

2 Dummy Load Method for Beams

2.1 Derivation for Rotations

The **dummy load method for beams** makes use of an equation we saw earlier. Slightly rewritten, this equation was

$$U = \int_{beam} \frac{M^2}{2EI} dx, \tag{2.1}$$

where we integrate over the entire beam. Just like in the dummy load method, we need to differentiate U. But now we differentiate with respect to a moment T. What we find is the rotation θ in the direction of T. So we get

$$\theta = \frac{\partial U}{\partial T} = \int_{beam} \frac{M \frac{\partial M}{\partial T}}{EI} dx = \int_{beam} \frac{Mm}{EI} dx, \tag{2.2}$$

where we have defined m as $\frac{\partial M}{\partial T}$.

2.2 Derivation for Displacements

We can also find displacements with this method. In that case we shouldn't differentiate U with respect to a moment T, but with respect to a force P. We then get

$$\delta = \frac{\partial U}{\partial P} = \int_{beam} \frac{M \frac{\partial M}{\partial P}}{EI} dx = \int_{beam} \frac{Mm}{EI} dx. \tag{2.3}$$

Note that m is now defined as $\frac{\partial M}{\partial P}$. This displacement is in the direction of the force P.

2.3 Using the Method

Now let's take a look at how to use this method. We have a beam and want to find the displacement at some point B. First we need to find $M(x)$. This is simply given by the moment diagram over the beam, caused by all the external forces.

We then need to find $m(x) = \frac{\partial M}{\partial P}$. It can be shown that the moment M depends linearly on P. So $M = mP$. To find m, we need to set $P = 1$. So we apply a unit load P at point B, perpendicular to the beam. We then derive the moment diagram, and we've got m.

Now all that is left for us to do is to apply the integral given by equation 2.3. That gives us the displacement we were looking for.

2.4 Avoiding the Integral

The integral in equation 2.3 can sometimes be very hard to evaluate. Therefore it is often allowed to make an approximation. For that, we split the beam up in a number of n segments. For every segment, we calculate the average values $M_{i_{ave}}$ and $m_{i_{ave}}$ using

$$M_{i_{ave}} = \frac{M_{i_{left}} + M_{i_{right}}}{2} \qquad \text{and} \qquad m_{i_{ave}} = \frac{m_{i_{left}} + m_{i_{right}}}{2}. \tag{2.4}$$

So what does this mean? We still need to find the moment diagrams for both M and m. Then, for every segment, we take the values for M on the left and right side of the segment, and take their average. In this way we find $M_{i_{ave}}$. We do the same for m to find $m_{i_{ave}}$.

In the end, when we have found all the average values, we simply apply

$$\delta = \sum \frac{M_{i_{mean}} m_{i_{mean}} L_i}{E_i I_i}. \tag{2.5}$$

Here L_i is the length segment i, and E_i and I_i are its E-modulus and moment of inertia. By the way, this equation also works to find the rotation θ. In that case the other definition of m needs to be applied. (The one with the unit moment.)

2.5 Beams and Bars

Sometimes a structure doesn't consist of only a bending beam. If the beam is supported by bars, then those bars deform as well. In that case the expression for U we have used earlier isn't complete. So (for simplicity of this example) let's suppose there's only 1 (vertical) bar supporting the (horizontal) beam. We then get

$$U = \int \frac{M^2}{2EI} dx + \frac{F^2 L}{2EA}. \tag{2.6}$$

Differentiating with respect to a load P now once more gives the displacement. We will find

$$\delta = \frac{\partial U}{\partial P} = \int \frac{Mm}{EI} dx + \frac{FfL}{EA}, \tag{2.7}$$

where the coefficient f is the force in the bar due to the applied unit load. If there are more beams or bars that deform, they also need to be considered. All parts that store energy need to be added to the above equation. It's as simple as that.

3 Statically Indeterminate Beams

3.1 Maxwell's theorem

Let's discuss statically indeterminate beams now. Statically indeterminate beams are usually difficult to analyze, as you need to use compatibility equations to solve the reaction forces. While finding these compatibility equations, the so-called **flexibility coefficients** can come in handy. The flexibility coefficient f_{BA} is the displacement of point B due to a unit load at point A. Now **Maxwell's Theorem** states that

$$f_{BA} = f_{AB}. \tag{3.1}$$

In words, the displacement of point B due to a unit load in A is the same as the displacement of A due to a unit load in B.

3.2 Other flexibility coefficients

It is also possible to calculate the flexibility coefficient m_{BA}, being the displacement of point B due to a unit moment at point A. Using these moment flexibility coefficients is identical as using the force flexibility coefficients, except that they involve moments and not forces.

Next to finding the displacement, you can also involve rotations in flexibility coefficients. For example, you can define the rotational flexibility coefficient f_{BA} as the rotation of a point B due to a unit force at point A. The same goes for unit moments.

3.3 Step 1 - Making the structure determinate

Suppose we have a statically indeterminate beam. To analyze the beam - finding all the reaction forces - we first have to remove supports until it becomes statically determinate. For every support, ask yourself: "If I remove it, will the structure be able to move?" If the answer is no, remove it.

3.4 Step 2 - Calculate displacements

Suppose we have removed supports at points $A_1, \ldots A_n$. We can now calculate the displacement ΔA_i of every point A_i for the new statically determinate beam, due to the external loads. The dummy load method for beams is suited for this rather well.

3.5 Step 3 - Calculate flexibility coefficients

Next to the displacements, we can also calculate the flexibility coefficients for the statically indeterminate beam. First remove all the external loads from the structure.

To find f_{AB} for some points A and B, just put a unit load at B and calculate the displacement at A. For the dummy load method for beams, you have to find $f_{A_iA_j}$ for every combination of i and j. So if n reaction forces are removed, you need to find n^2 flexibility coefficients (or about $\frac{1}{2}n^2$ is you use Maxwell's theorem to save time).

3.6 Step 4 - Formulate and solve compatibility equations

Now, using flexibility coefficients, several **compatibility equations** can be determined. For any node A_i, the total displacement in the original (statically indeterminate) structure is

$$\delta_{A_i} = \Delta A_i + R_{A_1} f_{A_iA_1} + R_{A_2} f_{A_iA_2} + \ldots + R_{A_n} f_{A_iA_n} = 0. \tag{3.2}$$

So we have n unknown reaction forces and n compatibility equations. All the reaction forces can be solved. And if all the reaction forces are known, it is relatively easy to calculate any displacement in the structure. For that, simply use the dummy load method for beams again.

4 Beams of Multiple Materials

4.1 Introduction to multiple material beams

Sometimes beams are made up out of layers of different materials. Usually each layer has a different E-modulus E. To be able to make calculations on these beams, we make a few assumptions. We assume that there is perfect bonding between the layers of material. So there can't be any slipping. We also assume that a line normal to the beam remans normal and perpendicular to the mid plane (MP) of the beam.

4.2 Weighted cross-sectional area

We want to be able to do calculations with beams consisting of multiple materials. So we define the weighted cross-sectional area A^* such that

$$dA^* = \frac{E}{E_{ref}} dA \qquad \Rightarrow \qquad A^* = \int_{A^*} dA^* = \int_A \frac{E}{E_{ref}} dA, \tag{4.1}$$

where E_{ref} is a reference E-modulus (usually taken to be the E-modulus of one of the materials in the beam). By examinig this definition, you see that stiff parts (with high E) contribute more to A^* than flexible parts.

4.3 Centroids and moment of inertia

The position of the weighted centroid can also be found by using the definition for the weighted area. The x-position of this centroid is

$$\bar{x}^* = \frac{1}{A^*}\int_{A^*} x\,dA^* = \frac{1}{A^*}\int_A x\frac{E}{E_{ref}}dA. \tag{4.2}$$

The y-position of the weighted centroid can be found identically. And once the position of the centroid is found, we can find the weighted moment of inertia I^*. The weighted moment of inertia about the x-axis is

$$I^* = \int_{A^*} y^2 dA^* = \int_A y^2\frac{E}{E_{ref}}dA, \tag{4.3}$$

where y is the vertical distance between the current point dA that is examined, and the position of the weighted centroid $\bar{y}^*$.

4.4 Stresses

The normal stress in a beam at some point (x, y) (with respect to the weighted centroid) due to a normal force P can be found using

$$\sigma(x, y) = \frac{P}{A^*}\frac{E(x, y)}{E_{ref}}. \tag{4.4}$$

When the beam is subject to a bending moment M, the stress at the point (x, y) can be found using

$$\sigma(x, y) = -\frac{My}{I^*}\frac{E(x, y)}{E_{ref}}. \tag{4.5}$$

Once more, the minus sign is present due to sign convention. When there is a combined normal force and bending moment, the above stresses can simply be added up. (By the way, the calculation of shear stress isn't different than for normal beams.)

4.5 Rotations and displacements

For beams of multiple materials, the forget-me-nots ought to be adjusted slightly. The new versions are

$$\theta = \frac{ML}{E_{ref}I^*}, \qquad \theta = \frac{PL^2}{2E_{ref}I^*}, \qquad \theta = \frac{qL^3}{6E_{ref}I^*}, \tag{4.6}$$

$$\delta = \frac{ML^2}{2E_{ref}I^*}, \qquad \delta = \frac{PL^3}{3E_{ref}I^*}, \qquad \delta = \frac{qL^4}{8E_{ref}I^*}. \tag{4.7}$$

4.6 Stiffness

For beams it's often interesting to know the stiffness. There are two kinds of stiffness. There is the **elongation stiffness** A and the **bending stiffness** D, defined as

$$A = \frac{N}{\epsilon}, \qquad \text{and} \qquad D = \frac{M}{\kappa}, \tag{4.8}$$

where κ is the curvature of the beam. If the beam consists of only layers of different materials, the values of A and D can be easily calculated. Let's suppose we have n layers $1 \ldots n$, each layer i, starting at z_{i-1}, ending at z_i and having E-modulus E_i. Here the values z are the vertical distance, measured from the centroid (with downward being negative). It can then be determined that

$$A = \sum_{k=1}^{n} E_k \left(z_k - z_{k-1}\right), \qquad \text{and} \qquad D = \sum_{k=1}^{n} E_k \left(\frac{z_k^3 - z_{k-1}^3}{3}\right). \tag{4.9}$$