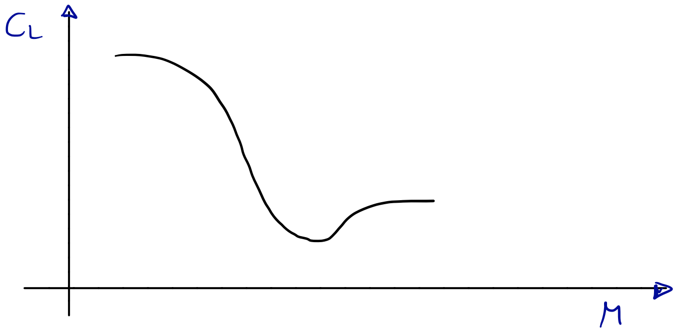
airbos

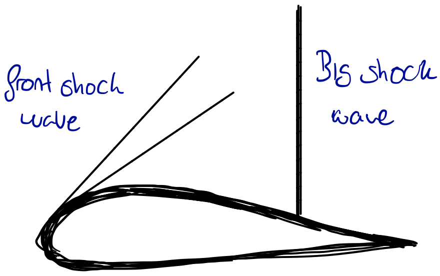
Lecture 1

Elements of gasdynamics
High speed flow
} Extra books

Practical wk 2.7 - 2.9 High speed



Lift coefficient drops due to the presence of shock waves: (separation)

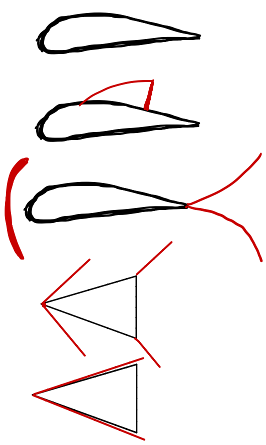


- → Big shock wave stablish transition between supersonic — subsonic.
- → Front shock waves make sure boundary layer is turbulent.
- → Big shock wave frequency can be dangerous for the wings

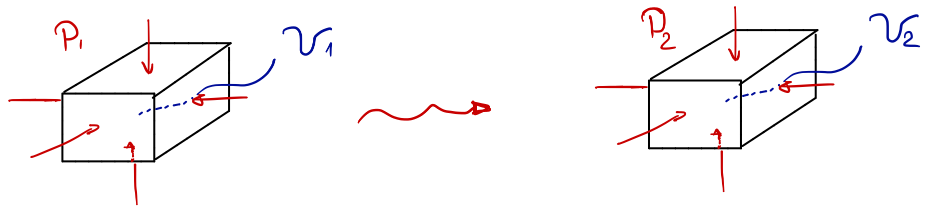
- For a turbine on an engine. To increase the efficiency the shock waves must be minimized.
- $\boxed{C_P \propto \frac{1}{\sqrt{1-M_\infty^2}}}$ Pressure is proportional to $\frac{1}{\sqrt{1-M_\infty^2}}$ Prandtl-glauert singularity

FLOWS

- SUBSONIC $M_\infty < 0.8$ →
- TRANSONIC $0.8 < M_\infty < 1$ →
- TRANSONIC $1 < M_\infty < 1.2$ →
- SUPERSONIC $M_\infty > 1.2$ →
- HYPERSONIC $M_\infty > 5$ →



COMPRESSIBLE



$P_2 > P_1$ $\quad$ $P_2 - P_1 = \Delta P$

$v_2 < v_1$ $\quad$ $v_2 - v_1 = \Delta v$

compressibility: $\boxed{\tau = -\frac{1}{v_1} \cdot \frac{\Delta v}{\Delta P}}$

Relationship between specific volume and density $\boxed{v = 1/\rho}$

$$\boxed{\tau = \frac{1}{\rho} \cdot \frac{d\rho}{dP}}$$

$$dv = d\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\, d\rho$$

Compressibility differs in the way you compress.

- constant temperature $\tau_T = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T$
- constant entropy $\tau_s = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_s$

- τ_{solids} $O(10^{-12})$
- τ_{liquid} $O(10^{-10})$ $\quad \frac{m^2}{N}$
- τ_{gas} $O(10^{-5})$

EQUATIONS FOR COMPRESSIBLE FLOW

two extra equations.

- ρ is an additional variable
- introduce energy balance.

variables: $P \quad \vec{V} \quad \rho \quad \underbrace{T \quad e}_{\text{added by the heat equation.}}$

$$\underbrace{\text{CONTINUITY, ENERGY,}}_{3}$$

$$\underbrace{\boxed{P = \rho R T} \quad \boxed{e = C_v \cdot T}}_{2}$$

GAS MODEL

ASSUMPTIONS

- 1-D flow
- Steady flow
- Inviscid flow
- No heat transfer
- No body forces.

CONTINUITY

$$\frac{\partial}{\partial t}\int_V \rho\, dV + \int_S \rho \vec{v} \cdot d\vec{s} = 0 \qquad \int_S \rho \vec{v} \cdot d\vec{s} = \int_{A_1} \rho \vec{v} \cdot d\vec{s} = \int_{A_2} \rho \vec{v} \cdot d\vec{s} = 0$$

$\boxed{\rho V A = \text{constant}}$ $\qquad$ $\boxed{\dot{m} = \text{constant}}$

MOMENTUM EQUATION

$$\frac{\partial}{\partial t}\int_V \rho\vec{V}\, dV + \int_S \rho\vec{V}(\vec{V}\cdot d\vec{S}) = -\int_S p\, d\vec{S} + \int_V \rho\vec{f}\, dV = \int_{A_1} \rho\vec{V}(\vec{V}\cdot d\vec{S}) + \int_{A_2} \rho\vec{V}(\vec{V}\cdot d\vec{S}) =$$

$$-\int_{A_1 \cup A_2} p\cdot d\vec{S} - \int_{\substack{\text{side}\\\text{surface}}} p\cdot d\vec{S} \longrightarrow -\rho_1 V_1^2\cdot A_1 + \rho_2 V_2^2\cdot A_2 = p_1 A_1 - p_2 A_2 - \int_1^2 p\cdot dA$$

$\boxed{p + \rho V^2 = \text{constant}}$ → for $A(x)$ constant. constant area.

ENERGY:

$$\frac{\partial}{\partial t}\int_V \rho\left(e + \tfrac{1}{2}V^2\right) dV + \int_S \rho\vec{V}\left(e + \tfrac{1}{2}V^2\right) d\vec{S} = \int_V \dot{q}\rho\, dV - \overbrace{\int_S p\vec{V}\cdot d\vec{S}}^{\text{work done by pressure forces.}} + \underbrace{\int_V \rho(\vec{f}\cdot\vec{V})\, dV}_{\text{body forces.}}$$

$$\int_S \rho\left(e + \frac{V^2}{2} + \frac{p}{\rho}\right)\vec{V}\cdot d\vec{S} = 0$$

$$\underbrace{\boxed{e + \frac{p}{\rho} + \frac{V^2}{2} = \text{constant}}}$$

$\boxed{h = e + \dfrac{p}{\rho}}$

$\boxed{H = \text{constant}}$

$\boxed{h = C_p\cdot T}$

$\boxed{H = h + \dfrac{V^2}{2}}$

$\boxed{\dot{m}_1 H_1 = \dot{m}_2 H_2}$

$\boxed{\rho\cdot H\cdot V\cdot A = \text{constant}}$

$E = e + \dfrac{V^2}{2}$ } total energy

$H = e + \dfrac{V^2}{2} + \dfrac{p}{\rho}$ } total enthalpy

Enthalpy is conserved. Energy is not conserved for the flow cause it can give away energy.

FOR A FLOW

STAGNATION CONDITIONS: conditions of the flow for velocity equals to 0

$\boxed{H = C_p\cdot T + \dfrac{V^2}{2}}$ → Temperature increases as the velocity decreases.

STATIC STATE: nonzero velocity, the flow goes from stagnation condition to static state.

STAGNATION TEMPERATURE IS CONSTANT DURING AN ADIABATIC PROCESS.

e_0, h_0 } are also constant. // Pressure is not known.

Lecture 2

Kinetic gas theory

- T and P are representations of molecular motion.
- T measure of average translational kinetic energy.
- P caused by impact of mollecules on the wall.

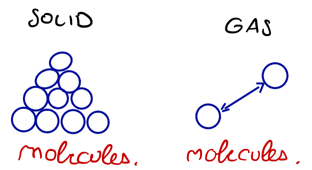
Perfect gas model

- Volume molecules neglected wrt vol
- Molecules are rigid spheres
- No forces between molecules

EQUATION OF STATE $\boxed{\frac{P}{\rho} = R \cdot T}$ $\quad R = \frac{\mathcal{R}}{\mu} =$

$\mathcal{R}$ → universal Gas constant
μ → molar mass

general case $\quad C_v = C_v(P, T) \rightarrow$ $\boxed{\text{constant for perfect gas}}$



$$C_v = \frac{\partial e}{\partial T}\bigg|_v$$ constant volume

Perfect gas model equations

$$\boxed{\rho = \frac{n \cdot M}{V}}$$

$$m \cdot \frac{d^2\vec{r}}{dt^2} = \vec{F}$$

m : mass of molecule
r : position vector
F : force acting on molecule

SCALAR PRODUCT $\quad \boxed{m\vec{r} \cdot \frac{d^2\vec{r}}{dt^2} = \vec{F} \cdot \vec{r}}$

Velocity of molecule

$$m \cdot \left[\frac{\partial}{\partial t} \cdot \vec{r} \cdot \frac{d\vec{r}}{dt} - \left[\frac{dr}{dt}\right]^2\right] = \vec{r} \cdot \vec{F} \qquad c = \frac{dr}{dt} \; \} \text{ speed}$$

$$m \cdot \left[\frac{d}{dt}\vec{r} \cdot \vec{c} - \vec{c}^{\,2}\right] = \vec{r} \cdot \vec{F} \; \} \text{ single molecule.}$$

ENSAMBLE AVERAGE → $\left[\bar{c}^2 = \frac{1}{N}\sum_{n=1}^{N} c_n^2\right]$ → STATIONARY SYSTEM $\quad$ time average

$$\boxed{\bar{C}^2 = \frac{1}{T}\int_0^T C_A^2 \, dt \,, \quad T \to \infty}$$

$$m \cdot \left[\underbrace{\frac{d}{dt} \cdot \vec{r} \cdot \vec{c}}_{0} - \vec{c}^{\,2}\right] = \underbrace{\vec{r} \cdot \vec{F}}_{} \; \} \; f(p) \text{ at the wall}$$

0 inside because interaction between molecules are random.

SPHERICAL VESSEL

Radius r_0
molecules N }

$$\underbrace{-N}_{\#\,mol} \cdot \underbrace{m \cdot \vec{C}^2}_{average} = \vec{F} \cdot \vec{r}$$

$\vec{F} = p \cdot A \cdot \vec{n} = p \cdot 4\pi r_0^2 \cdot \vec{n}$

$\vec{r} = -r_0 \vec{n}$

$|\vec{n}|^2 = 1 = \vec{n} \cdot \vec{n}$

$V = \frac{4}{3}\pi r_0^3$

$-N \cdot m \cdot \vec{C}^2 = -p \cdot 4\pi r_0^3$ → 3 comes from 2 + 1

$N \cdot m \cdot \vec{C}^2 = p \cdot 3V \longrightarrow \boxed{\frac{Nm}{V} \cdot \vec{C}^2 = 3p} \longrightarrow \boxed{\frac{p}{\rho} = \frac{1}{3}\vec{C}_2}$

$\frac{Nm}{V} \rightarrow \rho$

ONLY VALID FOR MONOATOMIC GASES

TRANSLATIONAL KINETIC ENERGY

$e_T = \frac{1}{2}\vec{C}^2 \qquad e = e_T = \frac{3}{2}RT$

$\frac{1}{3}\vec{C}_2$ must be equal to RT

$e_T = \frac{1}{2}\left(\vec{C}_x^2 + \vec{C}_y^2 + \vec{C}_z^2\right)$

$$\left[C_p = \frac{\partial h}{\partial T}\Big|_p = \frac{d\left(e + \frac{p}{\rho}\right)}{dT} = \frac{d(e + RT)}{dT} = C_v + R\right]$$

$C_v = \frac{3}{2}R$

$C_p = \frac{5}{2}R$

$$\boxed{\gamma = \frac{C_p}{C_v} = \frac{5}{3} = 1.\overline{66}} \qquad \boxed{C_p = C_v + R}$$

DIATOMIC GAS → $n_{DOF} = 5$

- HYPOTHESIS - EQUIPARTITION OF ENERGY

$e = e_T + e_R \longrightarrow 2\,\text{dof} \cdot \frac{1}{2}RT$

$\quad \hookrightarrow 3\,\text{dof} \cdot \frac{1}{2}RT$

$$\boxed{e = n_{dof} \cdot \frac{1}{2}RT}$$

$$\boxed{C_v = \frac{n}{2}R} \qquad \boxed{C_p = \frac{n+2}{2}R} \qquad \boxed{\gamma = \frac{n+2}{n}}$$

TRIATOMIC GAS

- $n_{DOF} = 6$, use above relationship.

VAN DER WAALS EOS

$$\left(P + \frac{c}{v^2}\right)(v + b) = RT$$

interactions between molecules.

volume of the molecules

accounts for volume of molecules and its interactions.

LIMITS FOR DOF

- High Temperature adds more degrees of freedom — become important after 600 K

$e = e_T + e_R + e_V + e_E + \ldots$

- High Pressures (low temperatures)

$v = \frac{RT}{P}$ $\quad$ $P\uparrow \; v = 0$, $T\downarrow \; v = 0$ } NOT POSSIBLE this goes to critical point.

AIR at

1 k $\longrightarrow \frac{C_v}{R} = -\frac{3}{2}$ $\longrightarrow$ translational

3 - 600 k $\rightarrow \frac{C_v}{R} = - 1.4 \rightarrow \boxed{\gamma = 1.4}$ $\longrightarrow$ trans + rotational

2000 k $\rightarrow \frac{C_v}{R} = -\frac{7}{2}$ $\longrightarrow$ trans + rot + vibration

$\frac{C_v}{R}$ increases with temperature.

AIR PROPERTIES

21% O2 $\longrightarrow \mu_O = 16$ g/mol

78% N2 $\longrightarrow \mu_N = 14$ g/mol

1% Ar $\longrightarrow \mu_A = 39.9$ g/mol

$\mu_{air} = 28.96$ g/mol

$R_{air} = \frac{R}{\mu_{air}} = 287$ J/kg K

DIATOMIC MOLECULES so $\longrightarrow$ $n = 5$

$\boxed{C_v = \frac{5}{2} R = 717.7 \text{ J/kg k}}$

$\boxed{C_p = \frac{7}{2} R = 1004.8 \text{ J/kg k}}$

SPEED OF SOUND

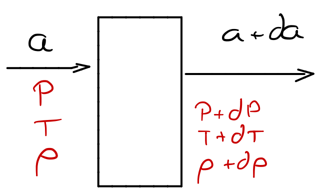
kinetic energy $\vec{C}^2$

velocity $= \sqrt{\vec{C}^2} > |\vec{C}| > a \longrightarrow \sqrt{\frac{8RT}{\pi}}$

its not only moving but also colliding

SOUND WAVE:

Follow soundwave as it moves. flow is moving towards us in the front of the sound wave and moving away behind the sound wave.



CONTINUITY 1

$\rho a = (\rho + d\rho)(a + da)$

$\boxed{a = -\rho \frac{da}{d\rho}}$

MOMENTUM 2

$p + \rho a^2 = (p + dp) + (\rho + d\rho)(a + da)^2$

$dp = -2a\rho da - a^2 d\rho$

$\boxed{\frac{da}{d\rho} = -\frac{a^2}{2a\rho} - \frac{1}{2a\rho} \cdot \frac{dp}{d\rho}}$

COMBINING 1 2

$a = \sqrt{\left(\frac{dp}{d\rho}\right)_s} \longrightarrow$ ISOTROPIC CHANGE OF STATE $\longrightarrow \frac{P_1}{P_2} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma} \longrightarrow \frac{P}{\rho} =$ constant

$\frac{dp}{d\rho} = \frac{d}{d\rho}(c\rho^{\gamma}) = \gamma c \rho^{\gamma - 1}$

$\frac{dp}{d\rho} = \frac{\gamma P}{\rho} = \gamma R T$

$\boxed{a = \sqrt{\gamma R T}}$

COMPRESSIBILITY

$$\tau = -\frac{1}{v}\cdot\frac{dv}{dp}\Big|_s = -\rho\cdot\frac{d\frac{1}{\rho}}{dp} = -\rho\left[\frac{0-\frac{\partial\rho}{\partial p}\big|_s}{\rho^2}\right]$$

$$\boxed{\tau = \frac{1}{\rho a^2}}$$

The lower the compressibility the higher the speed of sound.

MACH NUMBER

$$M = \frac{V}{a} = \frac{\text{convective speed}}{\text{sound prop. speed}}$$

$$M \rightarrow \frac{\overset{\text{ordered motion}}{\text{kinetic energy}}}{\underset{\text{molecular random motion}}{\text{internal energy}}} = \frac{V^2/2}{e} = \frac{V^2/2}{C_v T} = \frac{\gamma(\gamma-1)}{2}M^2$$

ENERGY EQUATION

Adiabatic flows: no heat added or taken

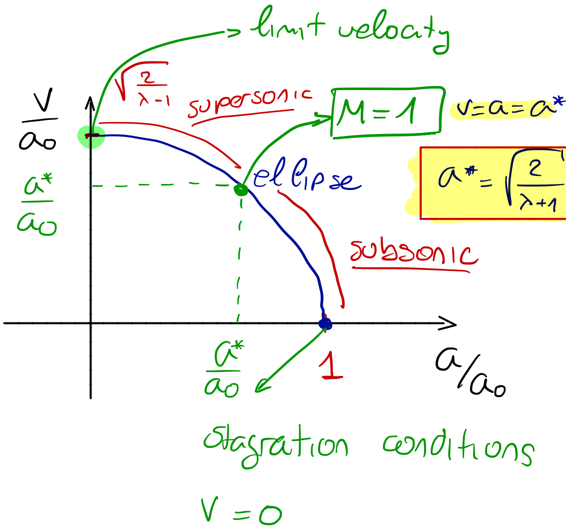
$$H = h_0 = h + \frac{1}{2}V^2 \longrightarrow \text{constant through the flow.}$$

$$C_p\cdot T_0 = C_p\cdot T + \frac{1}{2}V^2 \longrightarrow \text{write in terms of velocity.}$$

$$a = \sqrt{\gamma\cdot R\cdot T}$$

$$\frac{C_p}{\gamma R}\cdot a_0^2 = \frac{C_p}{\gamma R}a^2 + \frac{V^2}{2}$$

$$C_p = \frac{\gamma R}{\gamma-1} \longrightarrow \boxed{\frac{a_0^2}{\gamma-1} = \underbrace{\frac{a^2}{\gamma-1}}_{\text{internal + work energy}} + \underbrace{\frac{V^2}{2}}_{\text{kinetic energy}}} \longrightarrow \text{PLOT}$$



$$\boxed{a^* = \sqrt{\frac{2}{\gamma+1}}\,a_0}$$

LIMIT VELOCITY:

$$\boxed{V_{limit} = a_0\sqrt{\frac{2}{\gamma-1}}}$$

$$\boxed{V_{limit} = \sqrt{2C_pT_0}}$$

LIMIT MACH:

$$M\Big|_{V\to V_{limit}} = \frac{V_{limit}}{a_{limit}} \longrightarrow \infty$$

a^*, a_0 and V_{lim} are constant through the flow

ISENTROPIC RELATIONS

Using T_0, P_0, ρ_0

$$\boxed{\frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2} \qquad \boxed{\frac{P_0}{P} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}} \qquad \boxed{\frac{\rho_0}{\rho} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}}}$$

CONCORDE SKIN TEMPERATURES

What is the temperature of the nose?

$M = 2.06 \quad h = 16\ km \quad T = -56.5\ °C$

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2 = 1 + 0.2 \cdot (2.06)^2 = 1.85 \qquad T_0 = 1.85 \cdot 216.7 = 400k = 128\ °C$$

SONIC CONDITIONS

$$\frac{T^*}{T_0} = \frac{2}{\gamma + 1} \quad \Bigg\| \quad \frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \quad \Bigg\| \quad \frac{P^*}{P_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\boxed{(M^*)^2 = \frac{(\gamma+1) M^2}{2 + (\gamma-1) M^2}}$$

$$\boxed{M^2 = \frac{2}{\frac{(\gamma+1)}{(M^*)^2} - (\gamma - 1)}}$$

$$M^* = \frac{V}{a^*}$$

Relation between M and M^*

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{a^{*2}}{\gamma-1} + \frac{a^{*2}}{2}$$

If $M = 0 \longrightarrow M^* = 0$

If $M < 1 \longrightarrow M^* < 1$

If $M = 1 \longrightarrow M^* = 1$

If $M > 1 \longrightarrow M^* > 1$

If $M \to \infty \longrightarrow M^* \to \sqrt{\frac{\gamma+1}{\gamma-1}}$

Behaves the same way.

Relation between $\quad \frac{P_0}{P} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}}$ and $P_0 - P = \frac{1}{2} \rho V^2$

$$P_0 - P = P \left\{ \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}} - 1 \right\}$$

$$\left[\frac{1}{2} \rho V^2 = \frac{1}{2} \rho a^2 M^2 = \frac{1}{2} \gamma P M^2 \right]$$

$$f(M) = \frac{\left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}} - 1}{\frac{1}{2} \gamma M^2}$$

$$P_0 - P = \frac{1}{2} \rho V^2 \cdot \underbrace{f(M)}_{\text{compressibility factor}}$$

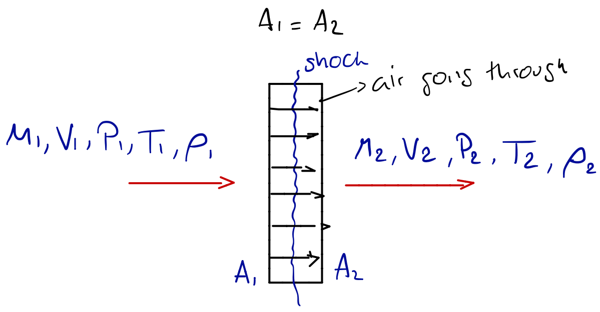
$$\boxed{P_0 - P = \frac{1}{2} \rho V^2 \left(1 + \frac{1}{4} M^2 + \frac{1}{40} M^4 + \dots\right)}$$

with this you can find the 0.3 Mach correction.

$$f(M) = 1 + \frac{1}{4} M^2 + \frac{1}{40} M^4 + \frac{1}{1660} M^6 + \dots$$

CALCULATION OF NSW PROPERTIES   NORMAL SHOCK WAVE

Obtain ratios of the thermodynamic properties ρ_2/ρ_1, P_2/P_1, T_2/T_1 across a NSW.



mass conservation

$$\rho_1 u_1 = \rho_2 u_2$$

momentum

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$$

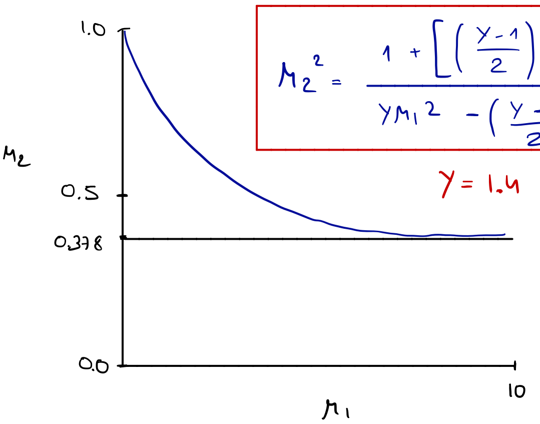
energy balance

$$\boxed{h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}} : \boxed{\frac{\gamma P_1}{\rho_1} + \frac{\gamma-1}{2} u_1^2 = \frac{\gamma P_2}{\rho_2} + \frac{\gamma-1}{2} u_2^2}$$

calorically perfect gas

- $h = C_p \cdot T$
- $P = \rho R T$ ideal gas law
- $C_p =$ constant

$$\boxed{M_2^2 = \frac{1 + \left[\left(\frac{\gamma-1}{2}\right) M_1^2\right]}{\gamma M_1^2 - \left(\frac{\gamma-1}{2}\right)}}$$



Using continuity: amount of mass constant $A_1 = A_2$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{u_1^2}{u_1 \cdot u_2} = \frac{u_1^2}{a^{*2}} = M_1^{*2}$$

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_1^2}{2 + (\gamma-1) M_1^2}}$$

Momentum equation: amount of momentum entering and in the flow must be equal to momentum leaving and staying.

$$P_2 - P_1 = \rho_1 u_1 - \rho_2 u_2 = \rho_1 u_1 (u_1 - u_2) = \rho_1 u_1^2 \left(1 - \frac{u_2}{u_1}\right) = \rho_1 u_1^2 \left(1 - \frac{\rho_1}{\rho_2}\right)$$

$$\frac{P_2}{P_1} - \frac{P_1}{P_1} = \frac{\rho_1}{P_1} u_1^2 \left(1 - \frac{\rho_1}{\rho_2}\right)$$

$$\frac{P_2}{P_1} = 1 + \frac{\gamma u_1^2}{\gamma R T_1} \left(1 - \frac{\rho_1}{\rho_2}\right) = 1 + \gamma M_1^2 \left(1 - \frac{\rho_1}{\rho_2}\right)$$

$$\boxed{\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1} (M_1^2 - 1)}$$

IDEAL GAS LAW $T_2/T_1 = \frac{P_2}{P_1} \cdot \frac{\rho_1}{\rho_2}$

$$\boxed{\frac{T_2}{T_1} = \frac{h_2}{h_1} = \left[1 + \frac{2\gamma}{\gamma+1} (M_1^2 - 1)\right] \cdot \frac{2 + (\gamma-1) M_1^2}{(\gamma+1) M_1^2}}$$

WHAT CAUSES THE SHOCK

$$\frac{u_2}{u_1} = \frac{2 + (\gamma-1) M_1^2}{(\gamma+1) M_1^2} = \frac{\rho_1}{\rho_2}$$

[strength is given by M_1]

$$\boxed{\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1} (M_1^2 - 1)}$$

2 solutions

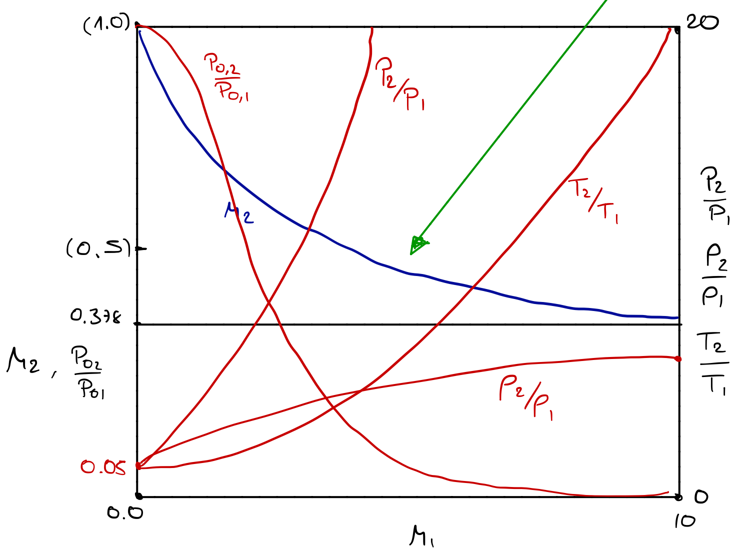
boundary conditions gives the needed sol so sometimes we have a shock and sometimes we don't.

MACH DOWNSTREAM

$$M_2^2 = \left(\frac{u_2}{a_2}\right)^2 = \underbrace{\left(\frac{u_2}{u_1}\right)^2}_{f(M_1,\gamma)} \cdot \underbrace{\left(\frac{u_1}{a_1}\right)^2}_{M_1^2} \cdot \underbrace{\left(\frac{a_1}{a_2}\right)^2}_{\frac{T_1}{T_2} \rightarrow f_2(M_1,\gamma)} = \boxed{M_2^2 = \frac{1+\left[\left(\frac{\gamma-1}{2}\right)M_1^2\right]}{\gamma M_1^2 - \left(\frac{\gamma-1}{2}\right)}}$$

IF $M_1 \rightarrow \infty$ we find:

$$\lim_{M_1\to\infty} M_2 = \sqrt{\frac{\gamma-1}{2\gamma}} = 0.378 \quad \Bigg\| \quad \lim_{M_1\to\infty} \frac{P_2}{P_1} = \infty \quad \Bigg\| \quad \lim_{M_1\to\infty} \frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1} = 6 \quad \Bigg\| \quad \lim_{M_1\to\infty} \frac{T_2}{T_1} = \infty$$



$$M_1^2 = \frac{\rho_1 u_1^2}{P_1}$$

PRANDT GARDLER

$$\frac{\gamma+1}{2(\gamma-1)} a^{*2} = \frac{a^2}{\gamma-1} + \frac{u^2}{2}$$

$a^* \propto a_0 = \sqrt{\gamma R T_0}$

a^* constant over NSW

$$\frac{u_2}{u_1} = \frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1^2} = \frac{1}{M_1^{*2}}$$

$$u_2 \cdot u_1 = \frac{u_1^2}{M_1^{*2}} = a_1^{*2} = a^{*2}$$

$$M_2^* = \frac{1}{M_1^*}$$

ENTROPY CHANGE ACROSS NSW

Shocks dont occur in subsonic flows. Entropy will be higher for a shock wave, because a shock wave will increase the randomness of the particles.

$$Tds = de + pdv \longrightarrow Tds = dh - vdp$$

$dh = de + d(pv) = de + pdv + vdp$

$de = dh - pdv - vdp$

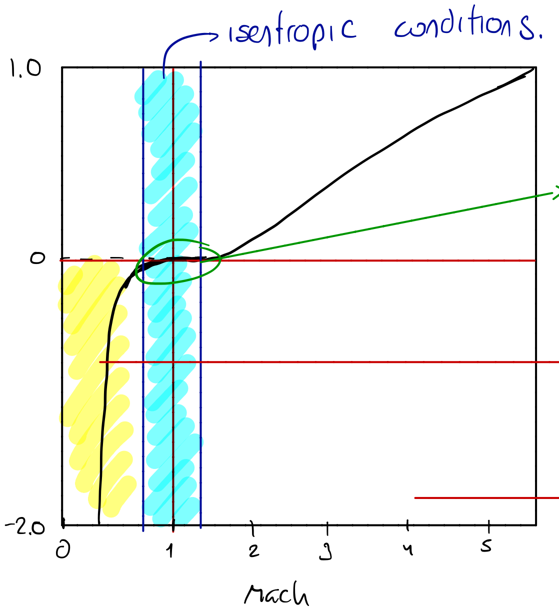
$v = \frac{1}{\rho} = \frac{RT}{P}$

$h = c_p T$

ENTROPY CONTINUED

$$ds = C_p \frac{dT}{T} - R\frac{dp}{p}$$ Integrate and set $$S_2 - S_1 = C_p \ln\left(\frac{T_2}{T_1}\right) - R\cdot\ln\left(\frac{P_2}{P_1}\right)$$

Can be substituted by lons equations.



At $M = 1$ $\quad f(M=1)=0$, $\quad f'(M=1)=0$, $\quad f''(M=1)=0$

SHOCK WAVE DEFINITION: Non-equilibrium thermodynamic process

- Large velocity gradiant causes viscous dissipation.
- Large temperature gradient causes heat diffusion.

NSW

$M_1 > 1 \longrightarrow$ $\qquad$ $M_2 < 1 \longrightarrow$

P_1, ρ_1, T_1 $\qquad$ P_2, ρ_2, T_2

How the properties of te flow change when crossing a Nomal shock wave.

TEMPERATURE

The total temperature is constant through a NSW

$$\left.\begin{aligned} C_p T_1 + \frac{u_1^2}{2} &= C_p T_{0,1} \\ C_p T_2 + \frac{u_1^2}{2} &= C_p T_{0,2} \end{aligned}\right\} \quad T_{0,2} = T_{0,1} \text{ because } C_pT_1 + \frac{u_1^2}{2} = C_pT_2 + \frac{u_2^2}{2}$$

PRESSURE:

The total prssure decreases across NSW in measure that the process becomes disipative.

$$S_2 - S_1 = C_p \ln\frac{T_2}{T_1} - R\ln\frac{P_2}{P_1} \quad \Bigg\| \quad S_{0,2} - S_{0,1} = C_p\cdot\ln\frac{T_{0,2}}{T_{0,1}} - R\cdot\ln\frac{P_{0,2}}{P_{0,1}}$$

$$\Delta s = f(\gamma, M_1)$$

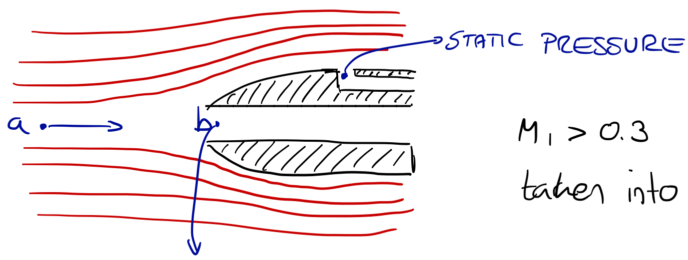
$$S_2 = S_{0,2} \qquad S_1 = S_{0,1}$$

$$S_2 - S_1 = -R\ln\left(\frac{P_{0,2}}{P_{0,1}}\right)$$

$$\boxed{\frac{P_{0,2}}{P_{0,1}} = f(\gamma, M_1)}$$

MEASUREMENT OF VELOCITY IN A COMPRESSIBLE FLOW:

The velocity can be determined through a pressure measurement.



stagnation pressure orifice.

$M_1 > 0.3$ compressibility taken into account.

For incompressible flows $M < 0.3$

$$V = \sqrt{\frac{2(P_0 - P)}{\rho}}$$

isentropic flow

$$\frac{P_{0,1}}{P_1} \stackrel{\text{IF}}{=} \left(1 + \frac{\gamma - 1}{2} M_1{}^2\right)^{\frac{\gamma}{\gamma-1}} \longrightarrow \text{solving for } M_1$$

$$M_1 = \sqrt{\frac{2}{\gamma-1}\left[\left(\frac{P_{0,1}}{P_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}$$

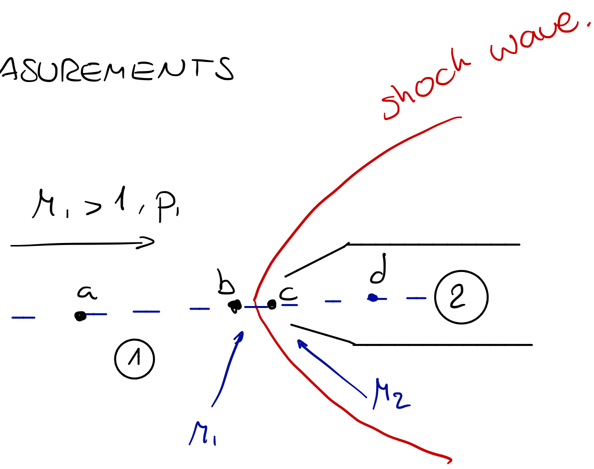
velocity can be obtained as

$$u_1 = M_1 \cdot a_1$$

$$u_1 = \sqrt{\frac{2a_1^2}{\gamma-1}\left[\left(\frac{P_{0,1}}{P_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]} \longrightarrow \text{We need to know } a_1 \text{ and } T_1.$$

SUPERSONIC FLOW MEASUREMENTS

a shock wave stands before the pitot tube.



$a \longrightarrow b$ the flow does not feel the probe

$b \longrightarrow c$ NSW deceleration $P_{0,2} \neq P_{0,1}$ and $\frac{P_2}{P_1}^{NSW} = f(\gamma, M_1)$

$c \longrightarrow d$ subsonic (isentropic deceleration)

$$\frac{P_{0,2}}{P_2} = \left(1 + \frac{\gamma-1}{2} M_2{}^2\right)^{\frac{\gamma}{\gamma-1}}$$

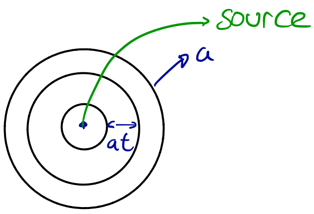
$$\boxed{\frac{P_{0,2}}{P_1} = \frac{P_{0,2}}{P_2} \cdot \frac{P_2}{P_1}}$$

RAYLEIGH PITOT FORMULA

$$\frac{P_{0,2}}{P_1} = \left[\frac{(\gamma+1)^2 M_1{}^2}{4\gamma M_1{}^2 - 2(\gamma-1)}\right]^{\gamma/\gamma-1} \cdot \frac{1-\gamma+2\gamma M_1{}^2}{\gamma+1}$$

Wave angle.

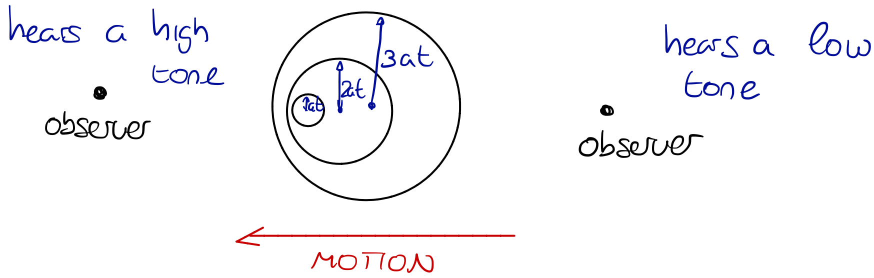
Consider a source of small disturbances in a quiescent medium.



Observer hears a frequency of $\boxed{f = \frac{1}{t}}$

Distance between wave front : $a \cdot t$

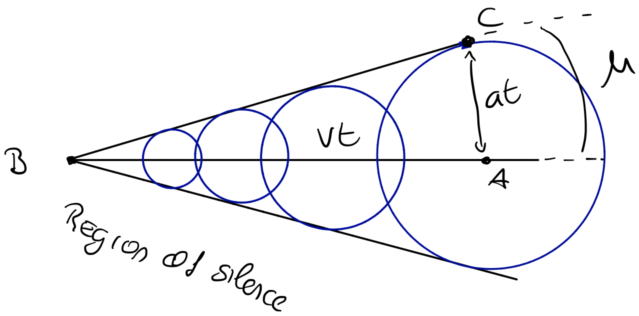
Consider the source in motion. → waves are not concentric anymore.



Doppler effect: difference in perceived frequency.

If $V < a$ the source is always inside the circles

Consider source moving faster than speed of sound.



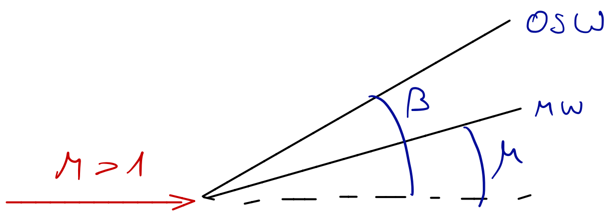
When the source moves from $A \to B$

the sound wave traveled from $A \to C$

$$\sin\mu = \frac{at}{vt} = \frac{a}{V} = \frac{1}{M}$$

$$\boxed{\mu = \arcsin\left(\frac{1}{M}\right)}$$

DISTURBANCES STRONGER THAT SOUND

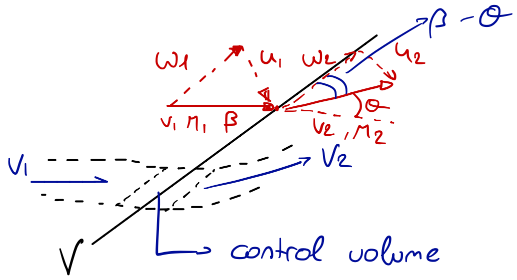


Shock wave angle: β

Mach wave angle: μ

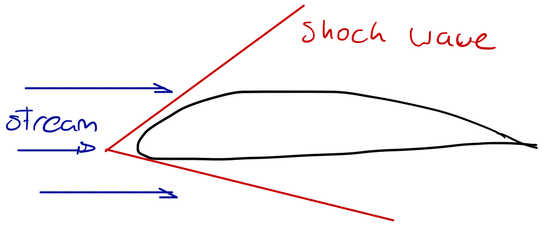
$\beta \geq \mu$

OBLIQUE SHOCK RELATIONS



For a NSW the shock wave is perpendicular to the flow stream

This is an oblique shock and happens for example when.



INTRODUCING THE MACH RELATIVE TO VELOCITY COMPONENTS

$$M_1 = \frac{W_1}{a_1} \;;\; M_{t2} = \frac{W_2}{a_2} \quad \Big\| \quad u_1 = V_1 \sin\beta \qquad u_2 = V_2 \sin(\beta-\theta)$$

$$\boxed{M_{n1} = \frac{u_1}{a_1} = \frac{V_1}{a_1}\sin\beta = M_1 \sin\beta} \qquad \boxed{M_{n2} = M_2 \sin(\beta-\theta)}$$

EQUATIONS FOR OSW

MOMENTUM:

tangential component

$$\int_S (\rho\vec{v}\cdot d\vec{s})w = -\int_S (p\cdot ds)_{tangential}.$$

But $d\vec{s}$ is perpendicular to the control surface $(Pds)_{tan}$ over the inlet and outlet surfaces is 0

$$-(\rho_1 A_1 u_1)W_1 + (\rho_2 A_2 u_2)W_2 = 0$$

$A_1 = A_2 \qquad \rho_1 u_1 = \rho_2 u_2$

$$\boxed{W_1 = W_2}$$

CONTINUITY:

$$\rho_1 V_1 \sin\beta = \rho_2 V_2 \sin(\beta-\theta)$$

$$\rho_1 u_1 = \rho_2 u_2$$

normal component

$$\int_S (\rho\vec{v}\cdot d\vec{s})u = -\int_S pds$$

$$-\rho_1 u_1 A_1 u_1 + \rho_2 u_2 A_2 u_2 = -(-P_1 A_1 + P_2 A_2)$$

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$$

$$\boxed{P + \rho u^2 = constant}$$

ENERGY EQUATION:

$$\int_s \rho\left(e+\frac{V^2}{2}\right)\vec{v}\cdot d\vec{s} = -\int_s p\vec{v}\cdot d\vec{s}$$

$A_1 = A_2$

$$-\rho_1\left(e_1+\frac{V_1^2}{2}\right)u_1\cancel{A_1} + \rho_2\left(e_2+\frac{V_2^2}{2}\right)u_2\cancel{A_2} = -\left(-p_1u_1\cancel{A_1}+p_2u_2\cancel{A_2}\right)$$

$$\cancel{\rho_1u_1}\left(\underbrace{e+\frac{p_1}{\rho_1}}_{h_1}+\frac{V_1^2}{2}\right) = \cancel{\rho_2u_2}\left(\underbrace{e_2+\frac{p_2}{\rho_2}}_{h_2}+\frac{V_2^2}{2}\right) \longrightarrow h_1+\frac{V_1^2}{2} = h_2+\frac{V_2^2}{2}$$

$$\boxed{V^2=u^2+w^2}$$

$$h_1+\frac{u_1^2+\cancel{w_1^2}}{2} = h_2+\frac{u_2^2+\cancel{w_2^2}}{2}$$

$$\boxed{h+\frac{u^2}{2} = \text{constant}}$$

SAME RELATIONS AS IN NSW

$$M_{n,2} = \frac{1+\frac{\gamma-1}{2}M_{n,1}^2}{\gamma M_{n,1}^2-\frac{\gamma-1}{2}} \qquad M_{n,2}=f(\gamma, M_{n,1}) \qquad M_{n,2}=f(\gamma, M_1, \beta)$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_{n,1}^2}{2+(\gamma-1)M_{n,1}^2} \longrightarrow f(\gamma, M_1, \beta)$$

$$\frac{p_2}{p_1} = f(\gamma, M_1, \beta) \qquad \frac{T_2}{T_1} = f(\gamma, M_1, \beta)$$

$$M_2 = \frac{M_{n,2}}{\sin(\beta-\theta)}$$

$$\underbrace{M_2 = f(\gamma, M_1, \beta, \theta)}$$

θ is not a variable, from geometry.

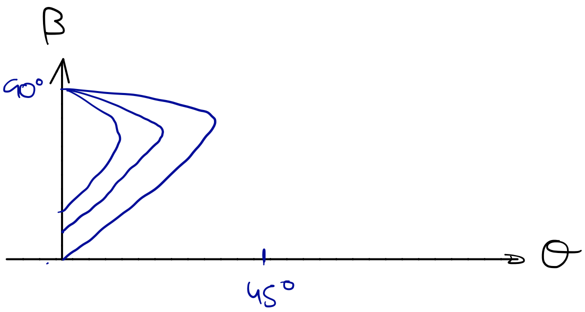
$$\tan\beta = \frac{u_1}{w_1} \qquad \tan(\beta-\theta) = \frac{u_2}{w_2}$$

$$\frac{\tan(\beta-\theta)}{\tan(\beta)} = \frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} = f(\gamma, M_1, \beta)$$

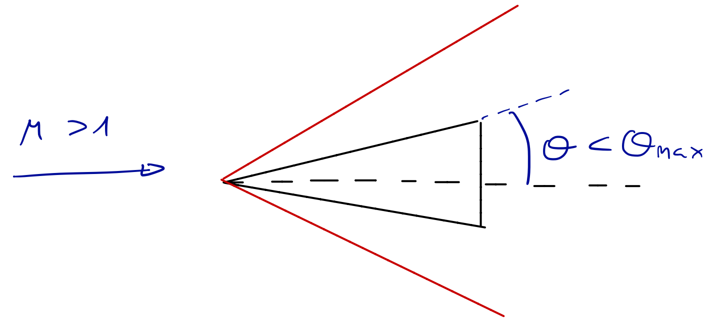
$M-\beta-\theta$ relation

$$\tan\theta = 2\cot(\beta)\,\frac{M_1^2\sin^2\beta-1}{M_1^2(\gamma+\cos(2\beta))+2}$$

$$\left.\begin{aligned}\theta &= f(\gamma, M_1, \beta)\\ M_2 &= f(\gamma, M_1, \beta)\end{aligned}\right\}\ \text{GRAPH } \theta-\beta$$

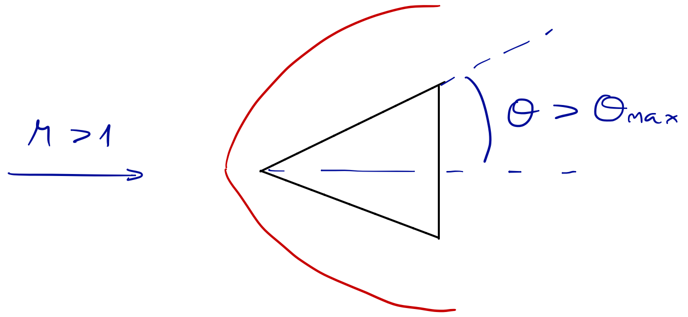


CONSIDERATIONS M–β–θ RELATION



If $\theta > \theta_{max} \rightarrow$ detached shock

locally the shock becomes normal.

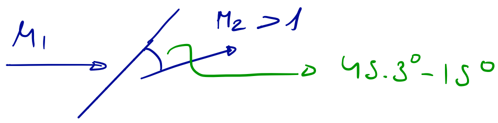


θ_{max} increases with the mach number

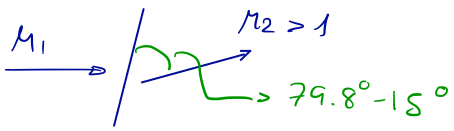
For any given couple M_1, θ with $\theta < \theta_{max}$ there are two possible solutions

EXAMPLE: $M_1 = 2.0$ $\theta = 15°$ $\rightarrow \begin{cases} \beta_1 = 45.3° \\ \beta_2 = 79.8° \end{cases}$

β_1 is a weak shock:



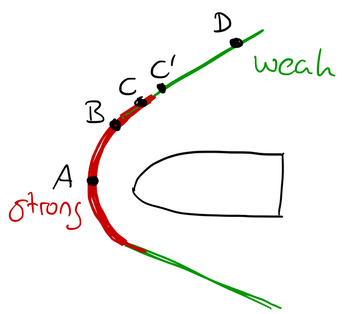
β_2 is a strong shock:

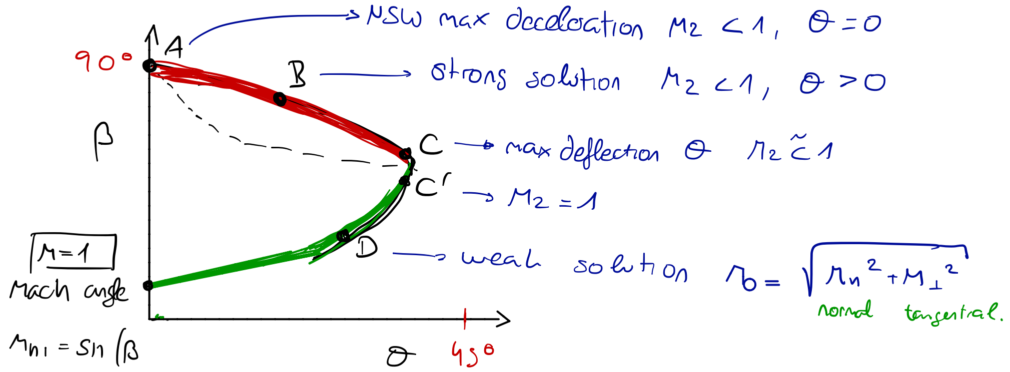


If $\theta = 0° \rightarrow$:

- stronger possible SW $\beta = 90°$
- weakest: (mach wave) $\beta = 0°$

DETACHED SW IN FRONT OF BLUNT BODY

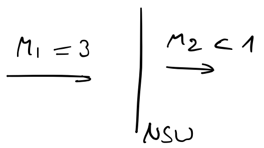




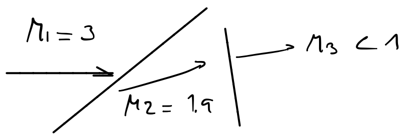
EXAMPLE SUPERSONIC INLET

- A Mach 3 flow is to be decelerated to subsonic regime (prior to combustion)
- 2 ways of deceleration
  - → NSW
  - → OSW + NSW

CASE 1



CASE 2

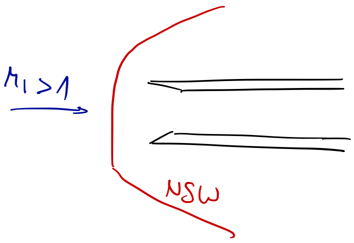


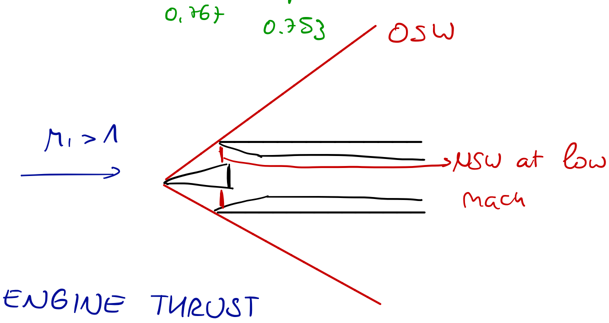
COMPARISON

$$\frac{\left(\frac{P_{0,3}}{P_{0,1}}\right)_2}{\left(\frac{P_{0,2}}{P_{0,1}}\right)_1} = 1.76$$

$$\frac{P_{0,2}}{P_{0,1}} = 0.328$$

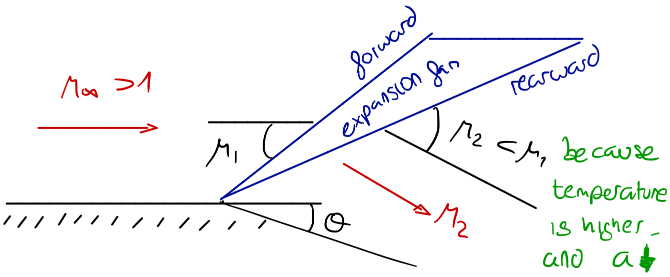
$$\frac{P_{0,3}}{P_{0,1}} = \underbrace{\frac{P_{0,3}}{P_{0,2}}}_{0.767} \cdot \underbrace{\frac{P_{0,2}}{P_{0,1}}}_{0.753} = 0.578$$





HIGHER DECELERATION EFFICIENCY → HIGHER ENGINE THRUST

PRANDTL-MEYER EXPANSION



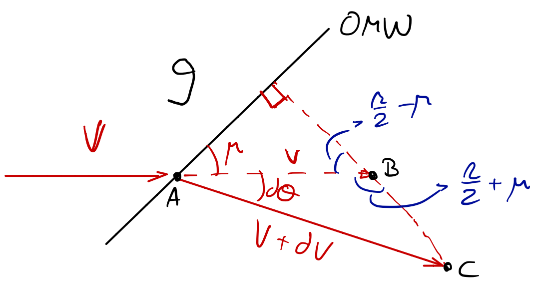
- supersonic stream
- expansion fan: an infinite number of Mach waves making continuous region
- Since the expansion is made out of Mach waves

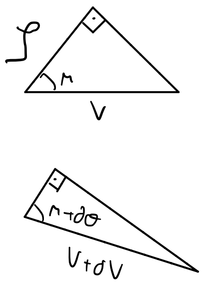
$ds = 0$ (isentropic)

$\boxed{\text{DETACHED NORMAL SHOCK WAVE HAPPENS WHEN } \theta \text{ is greater than } \theta_{max}}$

PROBLEM STATEMENT

Determine conditions ② from conditions ① and deflection θ





$$\frac{V + dV}{V} = \frac{\sin\left(\frac{\pi}{2} + \mu\right)}{\sin\left(\frac{\pi}{2} - \mu - d\theta\right)}$$

Derive:

$g = (V + dV)\cos(\mu + d\theta)$

$$\sin\left(\frac{\pi}{2} + \mu\right) = \sin\left(\frac{\pi}{2} - \mu\right) = \cos\mu \qquad \sin\left(\frac{\pi}{2} - \mu - d\theta\right) = \cos(\mu + d\theta) = \cos\mu\cos d\theta - \sin\mu\sin d\theta$$

$$1 + \frac{dV}{V} = \frac{\cos\mu}{\cos\mu\cos d\theta - \sin\mu\sin d\theta}$$

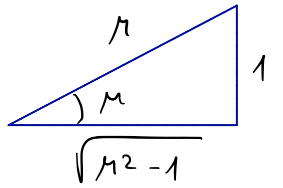
HYPOTHESIS: $d\theta \ll 1$

$$\begin{cases} \sin d\theta = d\theta \\ \cos d\theta = 1 \end{cases}$$

$$1+\frac{dV}{V}=\frac{\cos\mu}{\cos\mu-d\theta\sin\mu}=\frac{1}{1-d\theta\tan\mu}$$

Expand around $d\theta=0$ so we get

$$1+\frac{dV}{V}=1+d\theta\tan\mu \quad \text{or} \quad d\theta=\frac{dV/V}{\tan\mu}$$



$$\tan\mu=\frac{1}{\sqrt{M^2-1}}$$

THEREFORE

$$\boxed{d\theta=\sqrt{M^2-1}\,\frac{dV}{V}}$$

$$\left\{\boxed{\begin{array}{l} M>1 \\ d\theta>0 \\ dV>0 \end{array}}\right.$$

BY DIFFERENTIATION AND MANIPULATION

$$\frac{da}{a}=-\frac{\gamma-1}{2}M\left(1+\frac{\gamma-1}{2}M^2\right)^{-1}dM \qquad \frac{dV}{V}=\frac{1}{1+\frac{\gamma-1}{2}M^2}\cdot\frac{dM}{M}$$

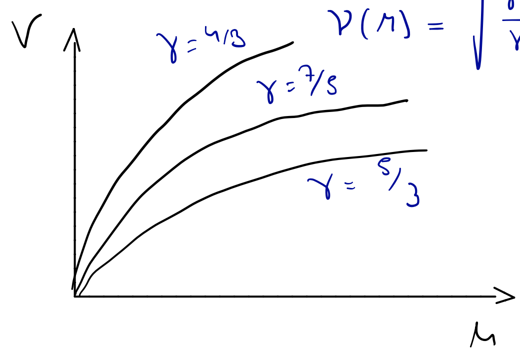
$$\boxed{\theta=\nu(M)=\int_{M_1}^{M_2}\frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\cdot\frac{dM}{M}}$$

$\longrightarrow$ INTEGRATING

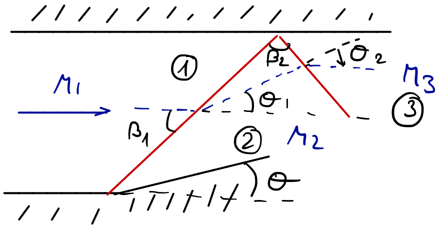
$$\nu(M)=\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)}-\tan^{-1}\left(\sqrt{M^2-1}\right)+C$$

can be found given that $\nu(M=1)=0$

$$\boxed{\theta=\nu M_2-\nu M_1}$$



SHOCK INTERACTIONS / REFLECTIONS



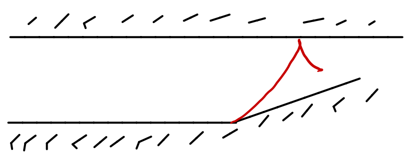
$\theta_1=\theta$

$\theta_2=-\theta_1$

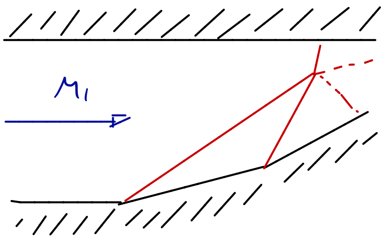
} The flow after the reflection must be parallel to the top wall

If M_1 is only slightly above the minimum for a straight OSW at given θ. The first shock may exist but the second is not possible. This situation creates the

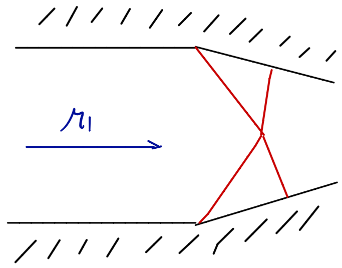
λ shock



SHOCK-SHOCK INTERACTION



Same family



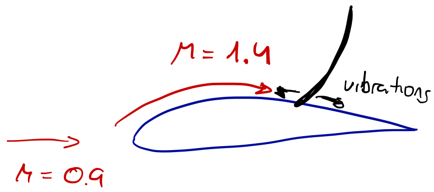
different family.

two oblique shocks that turn the shock in the same direction and when they interfere they form a new oblique shock.

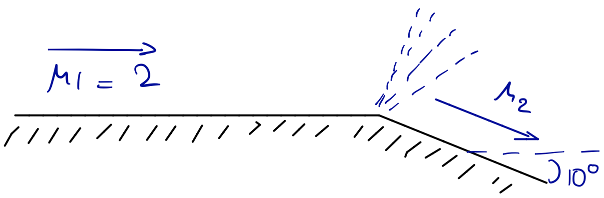
TOTAL PRESSURE LOSSES

APPLICATIONS

- suppersonic inlet
- swept wings



EXPANSION FAN EXAMPLE



$\gamma = 1.4$

$\theta = \nu(M_2) - \nu(M_1)$

$\nu(M_1) = 26.38°$

$\theta = 10 + 26.38° = 36.38 - \nu(M_2)$

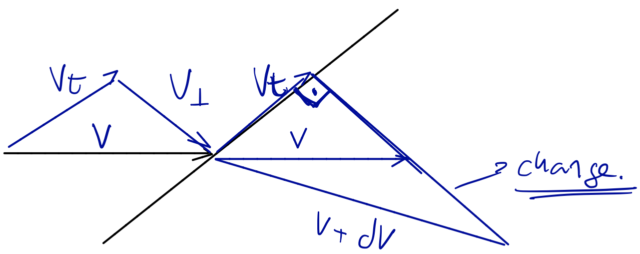
iterate $\longrightarrow M_2 = 2.4$

Pressure: $\dfrac{P_0}{P} = \left(1 + \dfrac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}}$

Isentropic: $P_{02} = P_{01}$

$$\frac{P_2}{P_1} = \left(\frac{1 + \frac{\gamma-1}{2} M_1^2}{1 + \frac{\gamma-1}{2} M_2^2}\right)^{\frac{\gamma}{\gamma-1}} = 0.55$$

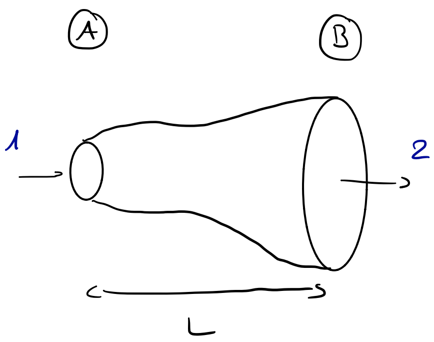
oblique waves dont modify tangential velocity



CHAPTER 10 COMPRESSIBLE FLOWS THROUGH NOZZLES

NOZZLE: converts potential / internal energy into kinetic energy. $u\uparrow$ $P\downarrow$

Diffusor: converts kinetic energy into potential energy $u\downarrow$ $P\uparrow$



$A: A(x)$, $u(x)$, $P(x)$, $M(x)$

Assume
- Steady flow $\frac{\partial}{\partial t} = 0$
- inviscid
- Adiabatic
- quasi one dimensional $u = V_x$ large → neglect the others.
  - → $A(x) = A$ $\quad \frac{dA}{dx} << \frac{L}{A}$ cross section varies smoothly.

CONTINUITY EQUATION

$$\rho_1 A_1 \cdot u_1 = \rho_2 A_2 u_2$$

$$\rho \cdot A \cdot u = \text{constant}$$

1 $\quad d(\rho A \cdot u) = 0$

2 $\quad u \cdot A \cdot d\rho + u \cdot dA \cdot \rho + du \cdot A \cdot \rho = 0$

3 $\quad \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{du}{u} = 0$

4 $\quad u^2 A d\rho + u^2 dA \rho + du \cdot u \cdot A \cdot \rho = 0$

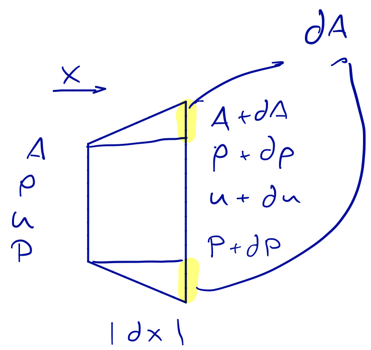
MOMENTUM EQUATION

$$A_1 P_1 + \rho_1 u_1^2 A_1 + \underbrace{\int_{A_1}^{A_2} P dA}_{\text{forces acting on wall.}} = A_2 P_2 + \rho_2 u_2^2 A_2$$

$$A_1 P_1 + \rho_1 \cdot u_1^2 \cdot A_1 + P_1 \cdot dA = \overbrace{(P + dP)(A + dA) + (\rho + d\rho)\underbrace{(u + du)^2}_{u^2 + 2udu + \cancel{du^2}^{\,0}}(A + dA)}^{\text{INSERT PHASE 2}}$$

$$\boxed{A dP + A u^2 d\rho + \rho u^2 dA + 2 \rho u A du = 0}$$

compare with 4 only the 2 is different.



$$\left(\frac{P + P + dP}{2}\right) dA$$

$$\boxed{P \cdot dA} + \frac{1}{2} \cancel{dP} \cdot dA$$

small

ENERGY EQUATION

total enthalpy = constant = $h + \frac{u^2}{2}$

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

$$A dP + \rho u A du = 0$$

$$\boxed{dP = -\rho u du} \quad \text{EULER'S EQUATION}$$

$$dP = -\rho u\, du$$

$$\frac{dP}{d\rho}\cdot\frac{d\rho}{\rho} = -u\, du$$

$$\left.\begin{matrix} \\ \end{matrix}\right\} a^2 = \frac{dP}{d\rho} \rightarrow a^2 \cdot \frac{d\rho}{\rho} = -u\, du$$

$$\frac{d\rho}{\rho} = -\frac{u^2}{a^2}\,\frac{du}{u} = \boxed{-M^2 \cdot \frac{du}{u}}$$

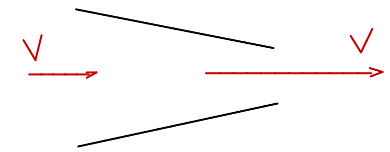
from eq ③

$$-\frac{du}{u} - \frac{dA}{A} = -M^2 \cdot \frac{du}{u} \longrightarrow \boxed{\frac{dA}{A} = (M^2-1)\,\frac{du}{u}}$$

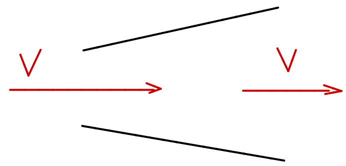
AREA–VELOCITY RELATION

CONSEQUENCES OF $\frac{dA}{A} = (M^2-1)\,\frac{du}{u}$

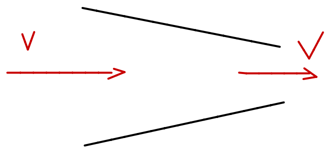
• $M<1 \quad dA<0$



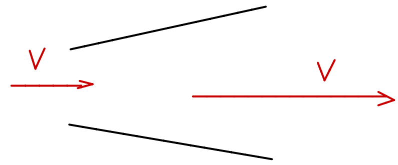
• $M<1 \quad dA>0$



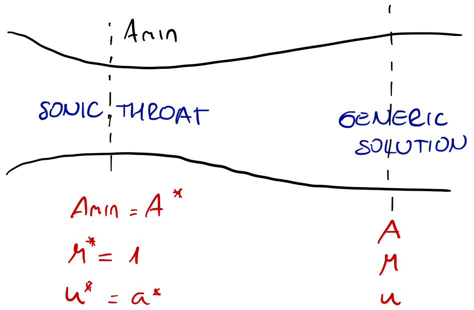
• $M>1 \quad dA<0$



• $M>1 \quad dA>0$



NOZZLE FLOWS: determine relation starting from the Velocity-Area relation.



FROM CONTINUITY:

$$\rho^* u^* A^* = \rho u A$$

$$\frac{A}{A^*} = \frac{\rho^* u^*}{\rho u} = \frac{\rho^* a^*}{\rho u} = \frac{\rho^*}{\rho_0}\,\frac{\rho_0}{\rho}\,\frac{a^*}{u}$$

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}$$

$$\frac{a^*}{u} = \frac{1}{M^*}$$

$$\frac{\rho_0}{\rho} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}}$$

$$(M^*)^2 = \frac{(\gamma+1)M^2}{2+(\gamma-1)M^2}$$

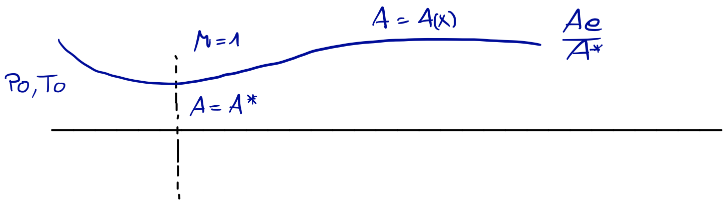
$$\boxed{\left(\frac{A}{A^*}\right)^2 \overset{IF}{=} \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{\gamma-1}}}$$

$$M = M\left(\frac{A}{A^*}\right) \qquad \frac{A}{A^*} = f(M)$$

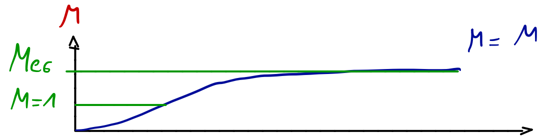
$$\left\{\begin{matrix} 1.)\ M<1 \\ 2.)\ M>1 \end{matrix}\right\}$$ solution depends of boundary condition: pressure ratio across the nozzle.

look into the table for IF $\rightarrow \frac{A}{A^*} = 2$ $\rightarrow$ $M = 0.31$, $M = 2.2$

CONSIDER A LAVAL NOZZLE

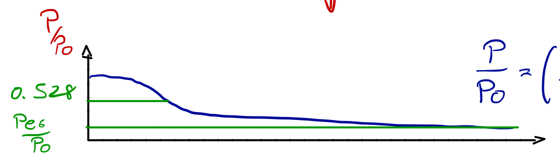


ASSUMING SUPERSONIC EXIT

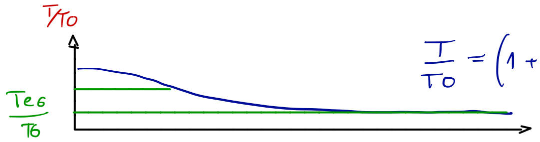


$M = M\left(\frac{A}{A^*}\right)$

↓ ISENTROPIC FLOW



$\frac{P}{Po} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{\gamma}{\gamma - 1}}$



$\frac{T}{To} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-1}$

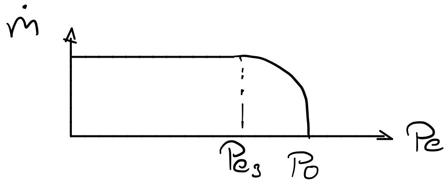
→ In an isentropic flow Pe must be the same Pe_6

→ If $Pe \neq Pe_6$

When $Pe = Pe_3$ → $At = A^*$

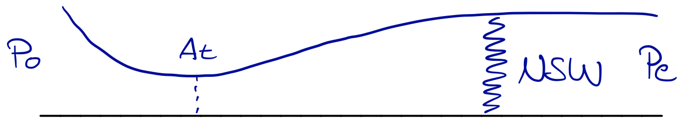
$\dot{m} = \rho^* u^* A^* = \rho^* u^* At$

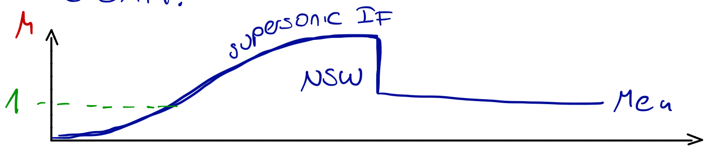
If $Pe < Pe_3$ → still the throat conditions are sonic $\dot{m}$ is fixed → choked mass flow regime.

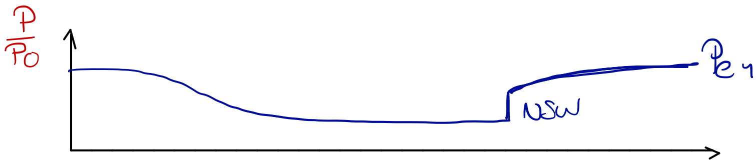


WHEN $Pe < Pe_3$ AND $Pe > Pe_6$

NSW is formed inside the nozzle or at its exit.





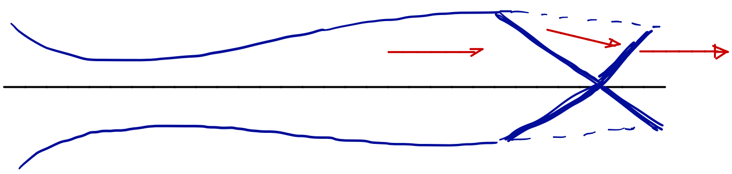


When Pe further decreases:

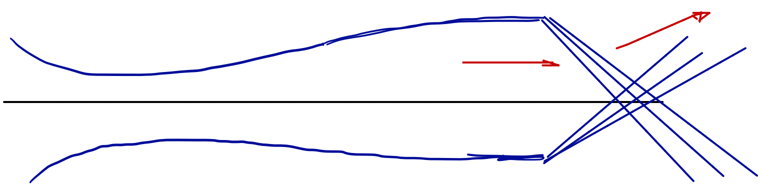
→ NSW travels downstream

We call Pe_5 the pressure where a NSW is placed at the nozzle exit.

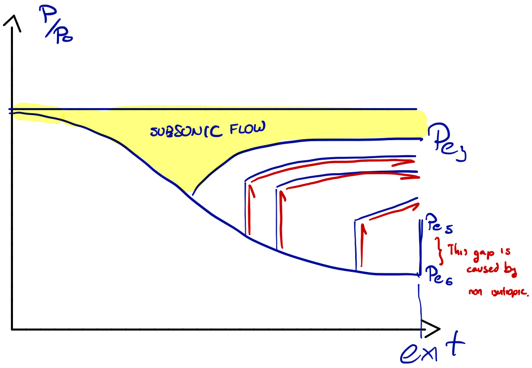
- If $Pe_6 < Pe < Pe_5$ a NSW is too strong and OSW appear. over-expanded flow regime.



- If $Pe < Pe_6$ the flow needs to expand further to adapt the pressure Prandtl-Meyer expansion fans appear. under-expanded flow regime.



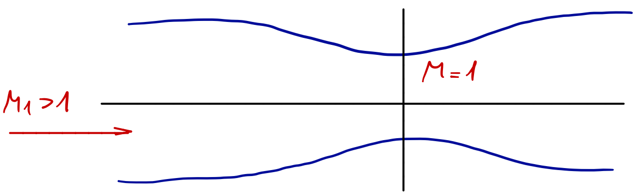
This are caused to accomodate the pressure of the jet to the ambient pressure.



DIFFUSERS

- Any duct designed to slow down the flow to lower velocity is a diffuser.
- The objective of the application of diffusers is to recover as high as possible total pressure in the stream.

$\longrightarrow$ The ideal diffuser performs an isentropic compression. $\Delta s = 0 \rightarrow P_{0,2} = P_{0,1}$



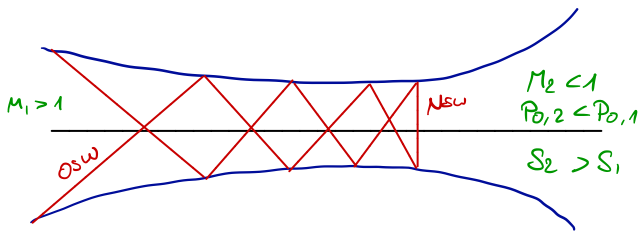
$M_2 < 1$
$P_{0,2} = P_{0,1}$
$S_2 = S_1$

} ISENTROPIC IS NOT REALISTIC DUE TO SW and viscous effects.

IDEAL: inverted isentropic compression.

Because V_g and V_x are definite we will have oblique shock wave.

ACTUAL SUPERSONIC DIFFUSER



} Due to entropy increase across SWs At in the real diffuser is larger than A^*

Worst case consider a NSW

Design Rule: assume a NSW

- compute Δs
- compute $P_{0,2}$

$\dot m = u_t \rho_t A_t$

$\rightarrow M = 1 \rightarrow a^* \cdot \rho^* \cdot A^*$

$\dot m = \frac{\sqrt{\gamma}}{\sqrt{RT}} P^* A^*$

$\dot m = C \cdot \frac{P_0}{\sqrt{T_0}} A^*$

$C = f(\gamma, R)$

decelerate from supersonic

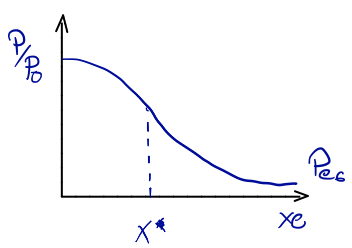
SUPERSONIC WIND TUNNELS

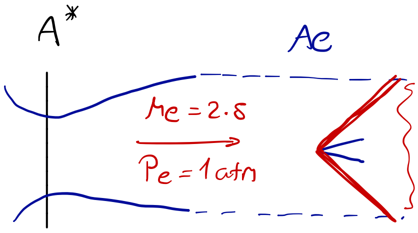
Objective: generate a uniform supersonic stream in a laboratory

E.g. $M = 2.5$

$\longrightarrow$ 1st choice:

Take a convergent-divergent nozzle with $\frac{A_e}{A^*} = 2.64$. Stablish $\frac{P_0}{P_e} = 17.1$ to exit in the ambient at P_{es}

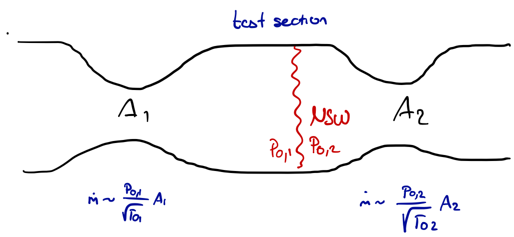




NOT EFFICIENT

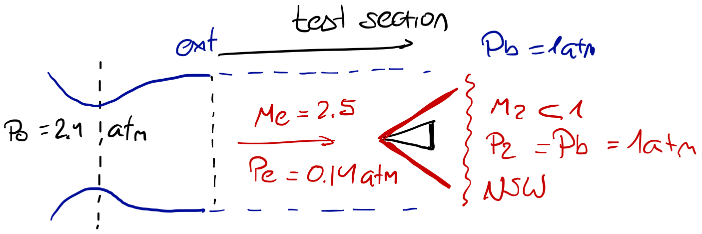
CONSEQUENCES

- need for air storage at $P_0 > 17$ atm
- large mass flow $\dot m = \rho^* u^* A^*$



$\frac{A_1}{A_2} = \frac{P_{0,2}}{P_{0,1}}$

→ 2nd CHOICE

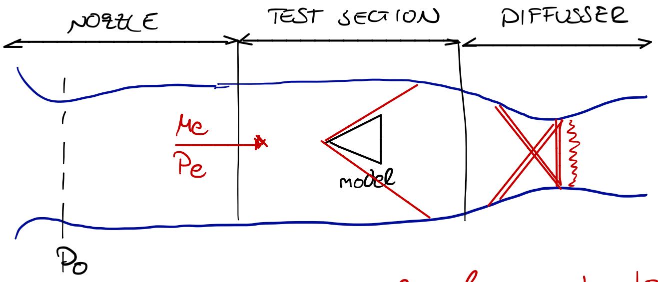


The nozzle exaust into a constant area duct.
Flow terminates with a NSW

CONSEQUENCES:

- $P_0 = 2.4$ needed reduction in mass flow.
- NSW acts as a diffuser low efficiency of deceleration.

→ 3RD CHOICE: LAVAL NOZZLE + DIFUSSER



Account for losses due to BL

CONSEQUENCES:

- Less entropy production decelerating with multiple OSWs
  → P_0 can be further lowered.
- Required pressure ratio P_e/P_0 can be obtained
  1.) Pressurized storage vessel ($P_0 > P_{atm}$)
  2.) Vacuum at the outlet ($P_0 = P_{atm}$)

DESIGN CRITERIA FOR A SUPERSONIC WIND TUNNEL

1. Operating Mach number M_e A_e/A^* — Given the size of the test section A_e the first throat A_1^* or A_{t1} is determined

2. Calculation of the section throat A_{t2} (diffuser) — Consider $\dot m = \rho u A = \rho_1^* u_1^* A_1^* = \rho_2^* u_2^* A_2^*$ assuming sonic conditions in both throats

$$\frac{A_{t2}}{A_{t1}} = \frac{\rho_1^* \cancel{u_1^*}}{\rho_2^* \cancel{u_2^*}} \overset{AF}{=} \frac{\rho_1^*}{\rho_2^*}$$ AF: adiabatic flow

$$\frac{A_{t2}}{A_{t1}} = \frac{P_1^* R T_2^*}{P_2^* R T_1^*} = \frac{P_1^*}{P_2^*} = \frac{P_{0,1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}}{P_{0,2}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}}$$

$$\boxed{\frac{A_{t2}}{A_{t1}} = \frac{P_{0,1}}{P_{0,2}}}$$

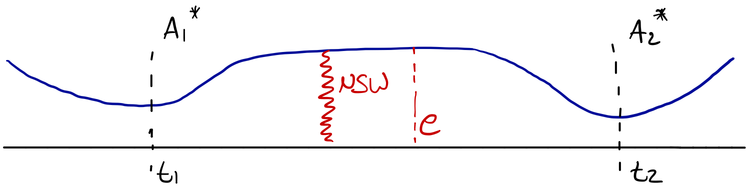
The second throat must always be larger than the first one.

In a preliminary design phase one may assume $\frac{P_{0,2}}{P_{0,1}}$ from a NSW at M_e

If $\frac{A_{t2}}{A_{t1}} < \frac{P_{0,1}}{P_{0,2}}$ → tunnel choking.

EXAMPLE Determine the SW position for a nozzle operating with P_{e5}, P_e, P_{e3}

Consider that the nozzle exit is followed by another duct with throat A_{t2} with sonic conditions.



$$\rho_1^* a_1^* A_1^* = \rho_2^* a_2^* A_2^*$$

$$p_0 = \rho_0 R T_0 \,;$$

$$\left(\frac{\rho^*}{\rho_0}\rho_0\sqrt{\gamma R}\sqrt{\frac{T^*}{T_0}}\sqrt{T_0}\,A^*\right)_1 = \left(\frac{\rho^*}{\rho_0}\rho_0\sqrt{\gamma R}\sqrt{\frac{T^*}{T_0}}\sqrt{T_0}\,A^*\right)_2$$

$$\boxed{\left(\frac{\rho^*}{\rho_0}\right)_1 = \left(\frac{\rho^*}{\rho_0}\right)_2} \qquad \boxed{\left(\frac{T^*}{T_0}\right)_1 = \left(\frac{T^*}{T_0}\right)_2}$$

$$\boxed{\frac{A_2^*}{A_1^*} = \frac{p_{0,1}}{p_{0,2}}\sqrt{\frac{T_{0,2}}{T_{0,1}}}} \xrightarrow{AF} \boxed{\frac{A_2^*}{A_1^*} = \frac{p_{0,1}}{p_{0,2}}}$$

$$M_e^2 \overset{AF}{=} \frac{1}{\gamma-1}\left[\left(1+2(\gamma-1)\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left(\frac{A^*}{A_e}\cdot\frac{p_0}{p_e}\right)^2\right)^{1/2}-1\right]$$

valid for isentropic and adiabatic flow

ADIABAIC NOZZLE FLOW

Even in presence of SWs we can obtain a relation returning the.

- Mach number M_e
- Useful in range $p_{e3} < p_e < p_{e6}$

CHAPTER 11 SUBSONIC COMPRESSIBLE FLOW OVER AIRFOILS

VELOCITY POTENTIAL EQUATION

Inviscid, compressible, subsonic flow around an object.

→ Irrotational flow

→ define a velocity potential $\quad \vec{V} = \vec{\nabla}\phi \quad$ or $\quad u = \frac{\partial\phi}{\partial x}, \; v = \frac{\partial\phi}{\partial y}$

Continuity equation:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \qquad \rho\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\rho}{\partial y} = 0$$

Eliminate ρ with momentum equation.

$$dp = -\rho V dV \longrightarrow dp = -\frac{\rho}{2}d(V^2) = -\frac{\rho}{2}d(u^2+v^2) = -\frac{\rho}{2}d\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]$$

$$\left.\frac{dp}{d\rho}\right|_s = a^2 \longrightarrow dp = a^2 \, d\rho$$

$$dp = -\frac{\rho}{2a^2}\, d\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]$$

$$\boxed{\frac{\partial \rho}{\partial x} = -\frac{\rho}{2a^2}\frac{\partial}{\partial x}\left[\;\right]}$$

$$\frac{\partial \rho}{\partial y} = -\frac{\rho}{2a^2}\frac{\partial}{\partial y}\left[\;\right]$$

$$\frac{\partial \rho}{\partial x} = -\frac{\rho}{a^2}\left(\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\,\partial y}\right)$$

SUBSTITUTING IN PREVIOUS RESULT

$$\left[1-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2} + \left[1-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2} - \frac{2}{a^2}\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial y}\right)\frac{\partial^2\phi}{\partial x\,\partial y} = 0$$

Using the energy equation we can eliminate the speed of sound.

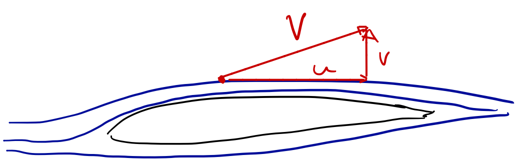
$$\boxed{a^2 = a_0^2 - \frac{\gamma-1}{2}V^2 = a_0^2 - \frac{\gamma-1}{2}(u^2+v^2) = a_0^2 - \frac{\gamma-1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]}$$

REMARKS

- Final result: a single PDE in terms of the unknown ϕ
- It can be solved for 2D shapes once given the B.C. at body sourface and ∞
- Once ϕ is known $\longrightarrow$ obtain u, v derivations
  - evaluate $a = a_0 f(\phi)$
  - calculate $M = V/a$
- Mostly solved numerically.
- Valid for any Mach Number.

LINEARIZED VELOCITY POTENTIAL

Consider a 2D body in uniform flow.



$$\left.\begin{aligned} u &= V_\infty + \hat{u} \\ v &= \hat{v} \end{aligned}\right\} \quad \phi = V_\infty x + \phi \quad \begin{cases} \dfrac{\partial\phi}{\partial x} = \hat{u} \\[2ex] \dfrac{\partial\phi}{\partial y} = \hat{v} \end{cases}$$

Therefore

SUBS ON V.P. Equation

$$\left.\begin{aligned} \frac{\partial\phi}{\partial x} &= V_\infty + \frac{\partial\phi}{\partial x} \qquad \frac{\partial\phi}{\partial y} = \frac{\partial\hat{\phi}}{\partial y} \end{aligned}\right\} \left[a^2 - \left(V_\infty + \frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2} + \left[a^2 - \left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2} - 2\left(V_\infty + \frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial y}\right)\frac{\partial^2\phi}{\partial x\,\partial y} = 0$$

PERTURBATION VELOCITY POTENTIAL EQUATION

$$\left[a^2-(V_\infty+\hat{u})^2\right]\frac{\partial \hat{u}}{\partial x}+\left[a^2-(\hat{v})^2\right]\frac{\partial \hat{v}}{\partial y}-2(V_\infty+\hat{u})(\hat{v})\frac{\partial \hat{u}}{\partial y}=0$$

$$\boxed{\frac{a_\infty^2}{\gamma-1}+\frac{V_\infty^2}{2}=\frac{a^2}{\gamma-1}+\frac{(V_\infty+\hat{u})^2+\hat{v}^2}{2}}$$

ENERGY EQUATION

$$\left(1-M_\infty^2\right)\frac{\partial \hat{u}}{\partial x}+\frac{\partial \hat{v}}{\partial y}=M_\infty^2\left[(\gamma+1)\frac{\hat{u}}{V_\infty}+\frac{\gamma+1}{2}\cdot\frac{\hat{u}^2}{V_\infty^2}+\frac{\gamma-1}{2}\frac{\hat{v}^2}{V_\infty^2}\right]\frac{\partial \hat{u}}{\partial x}+$$

$$+M_\infty^2\left[(\gamma+1)\frac{\hat{u}}{V_\infty}+\frac{\gamma+1}{2}\cdot\frac{\hat{v}^2}{V_\infty^2}+\frac{\gamma-1}{2}\frac{\hat{u}^2}{V_\infty^2}\right]\frac{\partial \hat{v}}{\partial x}+M_\infty^2\left[\frac{\hat{v}}{V_\infty}\left(1+\frac{\hat{u}}{V_\infty}\right)\left(\frac{\partial \hat{u}}{\partial x}+\frac{\partial \hat{v}}{\partial x}\right)\right]$$

SIMPLIFYING ASSUMPTIONS

CONSEQUENCES

Slender body, t and camber small
Small angle of attach

$$\left.\right\}\quad \frac{\hat{u}}{V_\infty}\ll 1\ ;\quad \frac{\hat{v}}{V_\infty}\ll 1$$

$$\frac{\hat{u}^2}{V_\infty^2}\ll\frac{\hat{u}}{V_\infty}$$

COMPARING FOR DIFFERENT TERMS

a) For $0\le M_\infty<0.8$ and $M_\infty\ge 1.2$

$$M_\infty^2\left[(\gamma+1)\frac{\hat{u}}{V_\infty}+\cdots\right]\frac{\partial \hat{u}}{\partial x}\ll\left(1-M_\infty^2\right)\frac{\partial \hat{u}}{\partial x}$$

b) For $M_\infty<5$

$$M_\infty^2\left[(\gamma-1)\frac{\hat{u}}{V_\infty}+\cdots\right]\frac{\partial \hat{v}}{\partial y}\ll\frac{\partial \hat{v}}{\partial y}$$ and also

$$M_\infty^2\left[\frac{\hat{v}}{V_\infty}\left(1+\frac{\hat{u}}{V_\infty}\left(\frac{\partial \hat{u}}{\partial x}+\frac{\partial \hat{v}}{\partial x}\right)\right]$$ is negligible.

VELOCITY PERTURBATION EQUATION

$$\boxed{\left(1-M_\infty^2\right)\frac{\partial \hat{u}}{\partial x}+\frac{\partial \hat{v}}{\partial y}=0}$$

VELOCITY POTENTIAL

$$\boxed{\left(1-M_\infty^2\right)\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0}$$

LINEARIZED EXPRESSION FOR PRESSURE COEFFICIENT

$$\boxed{C_p=-\frac{2\hat{u}}{V}}$$

The boundary conditions on the body surface and at infinity:

ϕ = constant at infinity because $\hat{u} = \hat{v} = 0$

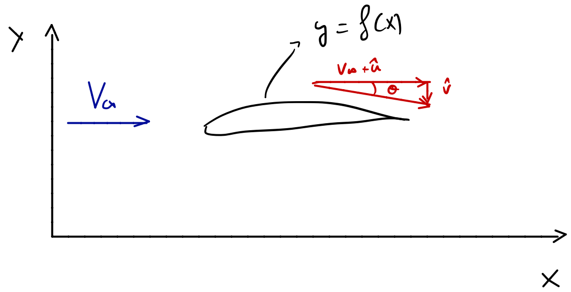
On the body the flow is tangent to the wall with slope θ

$$\tan\theta = \frac{v}{u} = \frac{\hat{v}}{V_\infty + \hat{u}} \approx \frac{\hat{v}}{V_\infty} = \frac{1}{V_\infty}\frac{\partial \hat{\phi}}{\partial y} \qquad \boxed{\frac{\partial \hat{\phi}}{\partial y} = V_\infty \tan\theta}$$

PRANDTL - GLAUERT COMPRESSIBILITY CORRECTION

Limited to thin airfoils at low α

Purely subsonic $M \leq 0.7$



Flow approximated by

$$(1 - M_\infty^2)\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0$$

Define $\beta = \sqrt{1 - M_\infty^2} \longrightarrow \beta^2 \frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0$

Apply coordinate transform:

$$\left.\begin{aligned} \xi &= x \\ \eta &= \beta y \end{aligned}\right\} \quad \bar{\phi}(\xi, \eta) = \beta \hat{\phi}(x, y)$$

$$\frac{\partial \hat{\phi}}{\partial x} = \frac{\partial \hat{\phi}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \hat{\phi}}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial \hat{\phi}}{\partial \xi}$$

$$\frac{\partial \hat{\phi}}{\partial y} = \frac{\partial \hat{\phi}}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \hat{\phi}}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial \hat{\phi}}{\partial \eta}$$

Recall that $\bar{\phi} = \beta \hat{\phi}$

$$\frac{\partial \hat{\phi}}{\partial x} = \frac{1}{\beta}\frac{\partial \bar{\phi}}{\partial \xi} \longrightarrow \frac{\partial^2 \phi}{\partial x^2} = \frac{1}{\beta}\frac{\partial^2 \phi}{\partial \xi^2}$$

$$\frac{\partial \hat{\phi}}{\partial y} = \frac{\partial \bar{\phi}}{\partial \eta} \longrightarrow \frac{\partial^2 \phi}{\partial y^2} = \beta \frac{\partial^2 \phi}{\partial \eta^2}$$

SUBSTITUTING IN STARTING EQUATION

$$\beta^2 \frac{1}{\beta}\frac{\partial^2 \phi}{\partial \xi^2} + \beta \frac{\partial^2 \bar{\phi}}{\partial \eta} = 0 \qquad \boxed{\frac{\partial^2 \bar{\phi}}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} = 0}$$

LAPLACE EQUATION

CONSEQUENCE:

We can obtain a linearized C_p from subsonic flow theory

$$C_p = -\frac{2\hat{u}}{V_\infty} = -\frac{2}{V_\infty}\frac{\partial \hat{\phi}}{\partial x} = -\frac{2}{V_\infty}\cdot\frac{1}{\beta}\cdot\frac{\partial \bar{\phi}}{\partial x} = -\frac{2}{V_\infty}\cdot\frac{1}{\beta}\cdot\frac{\partial \bar{\phi}}{\partial \xi}$$

INTRODUCE $\bar{u} = \dfrac{\partial \bar{\phi}}{\partial \xi}$

$C_p = \dfrac{1}{\beta}\left(-\dfrac{2\bar{u}}{V_\infty}\right) \qquad C_{p,0} = -\dfrac{2\bar{u}}{V_\infty} \qquad C_p = \dfrac{C_{p,0}}{\beta}$

$$\boxed{C_p = \frac{C_{p,0}}{\sqrt{1-M_\infty^2}}}$$

Prandtl - Glauert rule

Same works for C_l, C_m, C_d

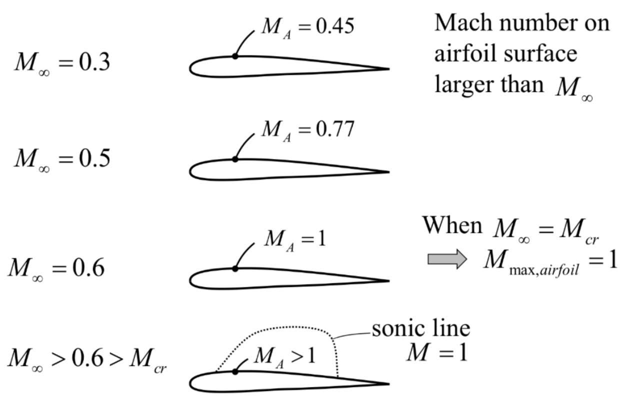
PROBLEM: If M_∞ is high enough ($M_\infty > M_{cr}$) local supersonic flow introduces wave drag and D'Alembert paradox will not be valid.

$$C_p = \frac{C_{p0}}{\sqrt{1-M_\infty^2} + \left[M_\infty^2 / \left(1+\sqrt{1-M_\infty^2}\right)\right] C_{p0}/2}$$

CRITICAL MACH NUMBER

If $(M_\infty^2 - 1) << 1$ linearized theory does not apply.

If $M_\infty > M_{cr}$ local supersonic flow large increase in drag coefficient



DETERMINATION OF CRITICAL PRESSURE COEFFICIENT.

Isentropic flow

minimum value static pressure on airfoil.

$$\frac{P_A}{P_\infty} = \frac{P_A}{P_0}\cdot\frac{P_0}{P_\infty} = \left[\frac{1+\frac{\gamma-1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}M_A^2}\right]^{\frac{\gamma}{\gamma-1}}$$

free stream static pressure

$$C_{p,A} = \frac{2}{\gamma M_\infty^2}\left(\frac{P_A}{P_\infty}-1\right) = \frac{2}{\gamma M_\infty^2}\left\{\left[\frac{1+\frac{\gamma-1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}M_A^2}\right]^{\frac{\gamma}{\gamma-1}}-1\right\}$$

If $M_\infty = M_{cr} \rightarrow M_A = 1$

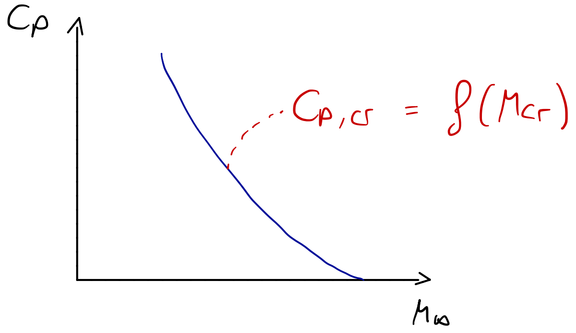
$M_A = 1$

$C_{p,crit}$ attained at minimum pressure

$$C_{p,A} = \frac{2}{\gamma M_\infty^2}\left(\frac{P_A}{P_\infty}-1\right) = \frac{2}{\gamma M_\infty^2}\left\{\left[\frac{1+\frac{\gamma-1}{2}M_\infty^2}{\frac{\gamma+1}{2}}\right]^{\frac{\gamma}{\gamma-1}}-1\right\}$$

$$C_{p,A} = \frac{2}{\gamma M_{cr}^2}\left\{\left[\frac{1+\frac{\gamma-1}{2}M_{cr}^2}{\frac{\gamma+1}{2}}\right]^{\frac{\gamma}{\gamma-1}}-1\right\}$$

$C_{p,cr}$ and M_{cr} are related graphically like

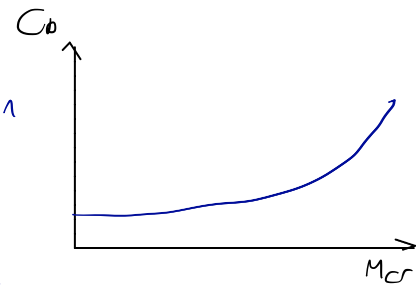


This result with the compressibility correction allows to estimate M_{cr} for a given airfoil.

PROCEDURE

1.) By experiment or theory obtain the value of minimum in the incompressible regime.

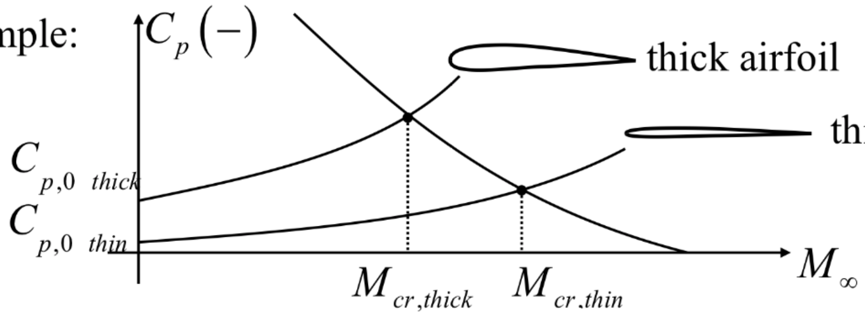
2.) Using a compressibility correction plot the function



3.) Determine intersection point where sonic conditions are achieved.

The result depends on the chosen airfoil

Example:



low speed

high speed

make sense.

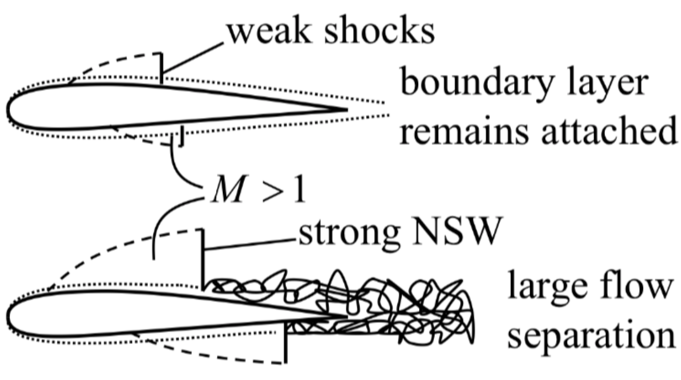
DRAG DIVERGENCE BARRIER

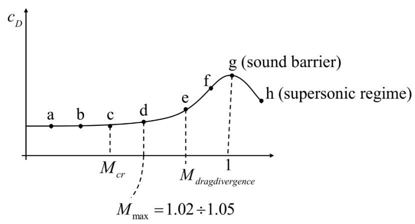
At $M_\infty \geq M_{cr}$ supersonic flow is terminated by shock waves.

of 196

- separation
- drag.

$M_\infty ; M_{cr}$



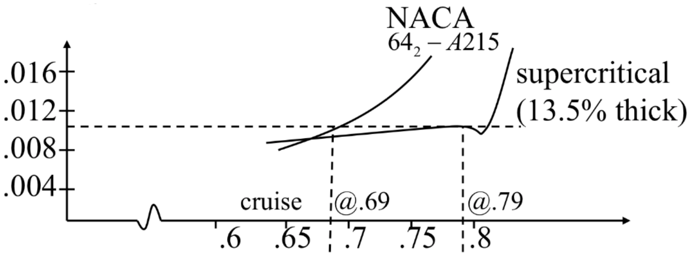


$M_\infty > M_{cr}$

SUPERCRITICAL AIRFOIL

Designed to obtain low drag increase in proximity of drag divergence.

- Higher cruise Mach number with moderate increase in fuel consumption



Achieved by a flat top.

CHAPTER 12 LINEARIZED SUPERSONIC FLOW

Starting from the linearized perturbation velocity potential

$$\left(1 - M_\infty^2\right) \frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0$$

Which holds for both subsonic and supersonic flow regimes.

- Supersonic $1 - M_\infty^2 < 0$ hyperbolic PDE
- Subsonic $1 - M_\infty^2 > 0$ elliptic PDE

For $M_\infty > 1$ $\longrightarrow$ define $\lambda = \sqrt{M_\infty^2 - 1}$

$$\lambda^2 \frac{\partial^2 \hat{\phi}}{\partial x^2} - \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0$$

Solution to this equation given by

$$\hat{\phi} = f(x - \lambda y)$$

Substitution:

$$\frac{\partial \hat{\phi}}{\partial x} = f'(x - \lambda y) \frac{\partial}{\partial x}(x - \lambda y) = f' \longrightarrow \frac{\partial^2 \hat{\phi}}{\partial x^2} = f''$$

Similarly: $\dfrac{\partial^2 \hat{\phi}}{\partial y^2} = \lambda^2 f''$

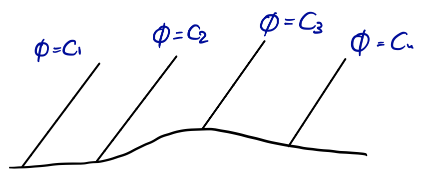
$$\lambda^2 \frac{\partial^2 \hat{\phi}}{\partial x} - \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0 \longrightarrow \lambda^2 f'' - \lambda^2 f'' = 0$$

Not specific!

The identity tells us that if $x - \lambda y = \text{constant}$

- $\hat{\phi}$ is also constant
- slope is $\frac{\partial y}{\partial x} = \frac{1}{\lambda} = \frac{1}{\sqrt{M_\infty^2 - 1}} = \tan\mu$

Flow properties are constant along lines locally inclined of μ. $\hat{\phi}$ is constant along Mach lines.



Waves of finite strength are not accounted for in linearized theory.

LINEARIZED C_p IN SUPERSONIC REGIME:

$$\left.\begin{aligned} \hat{u} &= \frac{\partial \hat{\phi}}{\partial x} = f' \\ \hat{v} &= \frac{\partial \hat{\phi}}{\partial y} = -\lambda f' \end{aligned}\right\} \quad \hat{u} = -\frac{\hat{v}}{\lambda}$$

Linearized boundary conditions

$$\hat{v} = V_\infty \tan\theta \approx V_\infty \cdot \theta$$

Small perturbations hypothesis

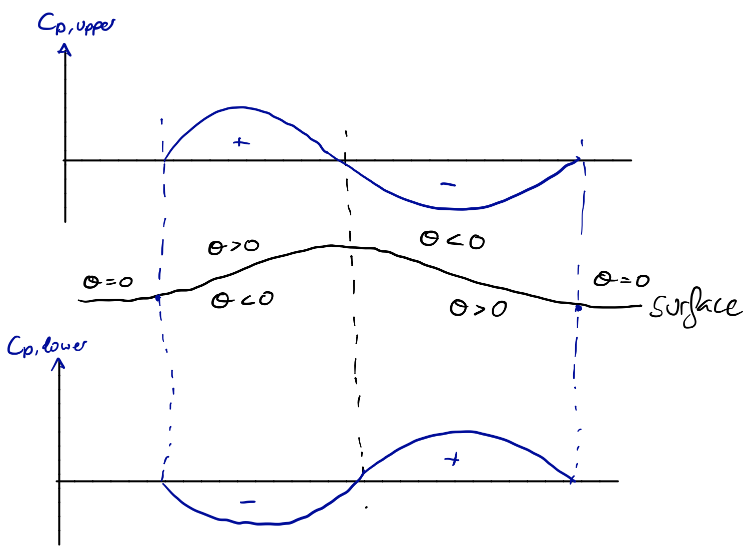
$$\hat{u} = -\frac{V_\infty \theta}{\lambda}$$

C_p is linearly proportional to local surface inclination.

$$C_p = -\frac{2\hat{u}}{V_\infty} = -\frac{2}{V_\infty}\left(-\frac{V_\infty \theta}{\lambda}\right)$$

$$C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}$$

Notice that in the C_p, $\theta > 0$ when measure above the horizontal.



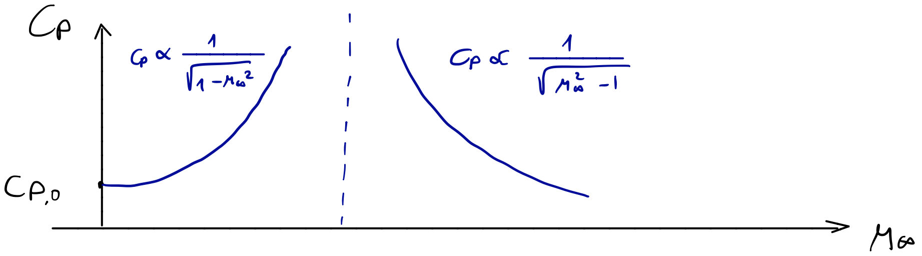
Pressure is higher on the top-front and bottom-rear segments.

- A net force will act on the profile.
  Wave drag characteristic of supersonic flows.
- Even without the presence of shocks, linearized theory predicts wave drag.

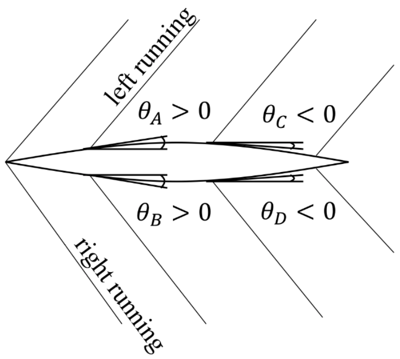
Supersonic Airfoils

Examining $C_p = \dfrac{2\theta}{\sqrt{M_\infty^2 - 1}}$ $\quad$ C_p decreases if M_∞ increases.

Combining this observation with subsonic compressible part.



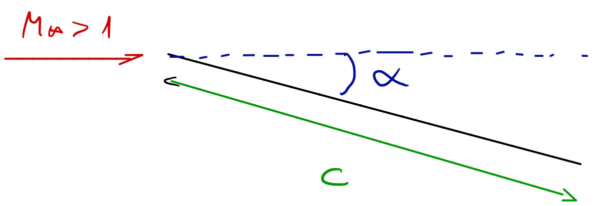
Sign convention for θ is different for left running and right running waves.



$$C_{p,A} = \frac{2\theta_A}{\sqrt{M_\infty^2 - 1}} \qquad C_{p,C} = \frac{2\theta_C}{\sqrt{M_\infty^2 - 1}}$$

$$C_{p,B} = \frac{2\theta_B}{\sqrt{M_\infty^2 - 1}} \qquad C_{p,D} = \frac{2\theta_D}{\sqrt{M_\infty^2 - 1}}$$

Flat Plate at Incidence



$$C_{p,lower} = \frac{2\alpha}{\sqrt{M_\infty^2 - 1}}$$

$$C_{p,upper} = \frac{-2\alpha}{\sqrt{M_\infty^2 - 1}}$$

$$C_n = \frac{1}{c}\int_0^c (C_{pl} - C_{pu})\, dx = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}} \cdot \frac{1}{c}\int_0^c dx = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}$$

$$C_a = \frac{1}{c}\int_{LE}^{TE} (C_{pu} - C_{pl})\, dy$$

Since the flat plate has zero thickness and camber

- dy is identically zero.
- $C_a = 0$ (pressure only normal to the axis)

$C_l = C_n \cos\alpha - C_a \sin\alpha$; $C_n - C_a \alpha$

$C_d = C_n \sin\alpha - C_a \cos\alpha$; $C_n \alpha - C_a$

For the flat plate case:

$C_l = C_n$

$C_d = C_n \alpha$

Lift coefficient $C_l = \dfrac{4\alpha}{\sqrt{M_\infty^2 - 1}}$

Wave drag coefficient $C_d = \dfrac{4\alpha^2}{\sqrt{M_\infty^2 - 1}}$

For a thin airfoil of arbitrary shape:

$C_l = \dfrac{4\alpha}{\sqrt{M_\infty^2 - 1}}$ still holds

$$\boxed{C_d = \frac{4}{\sqrt{M_\infty^2 - 1}} \cdot \left(\alpha^2 + g_c^2 + g_t^2\right)}$$

g_c is the function of the camber line

g_t is the function of the thickness