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Aerodynamics II

AE2210

p. 510 Perfect gases are those gases where the intermolecular spacing is so large, that intermolecular forces can be neglected. For a perfect gas, the following equation of state holds:

$$p = \rho R T$$

writing $\rho = \frac{1}{v}$ results in

$$p v = RT$$

p.519-520 The energy of a molecule is the sum of its translational, rotational, vibrational and electronic energies. Summing this energy of a volume filled with molecules yields the internal energy e of the gas. Enthalpy is defined as a function of e:

$$h = e + pv$$

Noting that ~~e and h~~ for a perfect gas, e and h only depend on temperature allows expressing these variables as functions of T and specific heats:

$$e = c_v(T)T$$

$$h = c_p(T)T$$

If c_v and c_p are constant, a gas is a calorically perfect gas. Defining the ratio of specific heats yields a few useful equations:

$$c_p - c_v = R$$

$$\gamma = \frac{c_p}{c_v}$$

$$c_p = \frac{\gamma R}{\gamma - 1} \quad \text{(constant volume process)}$$

$$c_v = \frac{R}{\gamma - 1} \quad \text{(constant pressure process)}$$

p.523-524 First Law of Thermodynamics

$$\delta q + \delta w = de$$

↳ State variable, hence exact differential de (rather than δe)

Many processes can deliver δq and δw. Three processes are most important.

1. Adiabatic p: $\delta q = 0$ (no heat transport)
2. Reversible p: no effects of viscosity, thermal conductivity and mass diffusion
3. Isentropic p: both adiabatic and reversible

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For a reversible process, the first law of thermodynamics modifies to:

$$\delta w = -p\,dv$$
$$\delta q - p\,dv = de$$

p.524-526 The first law doesn't define in which direction energy flows. Entropy s helps with that, as specified by the second law of thermodynamics

$$ds = \frac{\delta q}{T} + ds_{irreversible}, \quad ds_{irrev} > 0 \;\Big\}\; ds \geq \frac{\delta q}{T}$$
$$\geq 0 \quad \text{(adiabatic process)}$$

In words: entropy cannot be destroyed. Entropy is a state variable and is defined as

$$ds = \frac{\delta q_{rev}}{T}$$

Substituting this into the (modified) first law gives

$$T\,ds = de + p\,dv$$
$$T\,ds = dh - v\,dp \quad \text{(combined with definition of enthalpy)}$$

Combining these results with the expressions for e and h in terms of specific heats, the equation of state and integrating gives equations to compute the entropy increase:

$$ds = c_v \frac{dT}{T} + \frac{p\,dv}{T} = c_v \frac{dT}{T} + R\frac{dv}{v}$$
$$s_2 - s_1 = c_v \ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{v_2}{v_1}\right)$$
$$ds = c_p \frac{dT}{T} - \frac{v\,dp}{T} = c_p \frac{dT}{T} - R\frac{dp}{p}$$
$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right)$$

p.526-528 For an isentropic process, $s_2 - s_1 = 0$. This leads to the isentropic relations:

$$\ln\left(\frac{v_2}{v_1}\right) = -\frac{c_v}{R}\ln\left(\frac{T_2}{T_1}\right)$$
$$\frac{v_2}{v_1} = \left(\frac{T_2}{T_1}\right)^{-c_v/R} = \left(\frac{T_2}{T_1}\right)^{-(1/\gamma-1)}$$
$$\Rightarrow \frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{1/\gamma-1}$$
$$\ln\left(\frac{p_2}{p_1}\right) = \frac{c_p}{R}\ln\left(\frac{T_2}{T_1}\right)$$
$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{c_p/R} = \left(\frac{T_2}{T_1}\right)^{\gamma/\gamma-1}$$
$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\gamma/\gamma-1}$$

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p.530-531 The compressibility of a fluid element with volume v, ~~and~~ having a pressure p exerted on its walls is defined as:

$$\tau = -\frac{1}{v}\frac{dv}{dp} = \frac{1}{\rho}\frac{d\rho}{dp}$$

Physically, the compressibility is "the fractional change in volume of the fluid element per unit change in pressure".

$$\tau_T = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T \quad \text{(isothermal compressibility)}$$

$$\tau_S = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_s \quad \text{(isentropic compressibility)}$$

p.532-533 Whereas incompressible flow obeys purely mechanical laws, compressible flow (also) needs thermodynamic considerations. The primary variables are p, $\vec{V}$, ρ, e and T, which need 5 equations to be solved:

Continuity:

$$\frac{\partial}{\partial t}\iiint_V \rho\, dV + \oint_S \rho \vec{V}\cdot d\vec{S} = 0 \quad \text{(Eq. 2.48)}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\vec{V} = 0 \quad \text{(Eq. 2.52)}$$

Momentum:

$$\frac{\partial}{\partial t}\iiint_V \rho\vec{V}\, dV + \oint_S (\rho\vec{V}\cdot d\vec{S})\vec{V} = -\oint_S p\, d\vec{S} + \iiint_V \rho \vec{f}\, dV \quad \text{(Eq. 2.64)}$$

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho f_x \quad \text{(Eq. 2.113a)}$$

$$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho f_y \quad \text{(Eq. 2.113b)}$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho f_z \quad \text{(Eq. 2.113c)}$$

Energy:

$$\frac{\partial}{\partial t}\iiint_V \rho\left(e + \frac{V^2}{2}\right) dV + \oint_S \rho\left(e + \frac{V^2}{2}\right)\vec{V}\cdot d\vec{S} = \iiint_V \dot{q}\rho\, dV - \oint_S p\vec{V}\cdot d\vec{S} + \iiint_V \rho(\vec{f}\cdot\vec{V})\, dV \quad \text{(Eq. 2.95)}$$

$$\rho\frac{D}{Dt}\left(e + \frac{V^2}{2}\right) = \rho\dot{q} - \nabla\cdot p\vec{V} + \rho(\vec{f}\cdot\vec{V}) \quad \text{(Eq. 2.114)}$$

Equation of state:

$$p = \rho R T \quad \text{(Eq. 7.1)}$$

Internal energy:

$$e = c_v \cdot T \quad \text{(Eq. 7.6a)}$$

p.533-536 Static quantities are quantities felt when moving with the flow at local ~~a~~ flow velocity. When adiabatically brought to rest, the total ~~quantities are obtained.~~ temperature is obtained.

From the energy equation, it can be derived (p. 534-535) that

$$h + \frac{V^2}{2} = \text{constant} \quad \text{(along a streamline)}$$

From the definition of total ~~pressure~~, quantities ($V=0$) and the fact that the derivation of the equality above assumes adiabatic flow the constant can be determined to be the total enthalpy

$$h + \frac{V^2}{2} = \text{constant} = h_0 \quad \text{(along a streamline)}$$

If the gas is calorically constant,

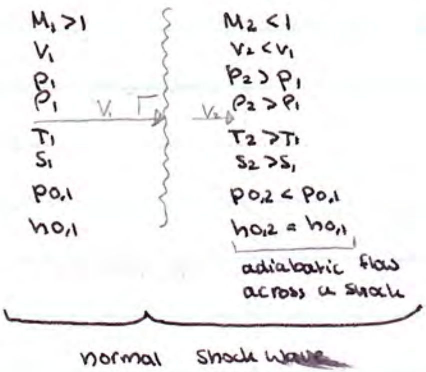
$h_0 = c_p \cdot T_0 = \text{constant}$

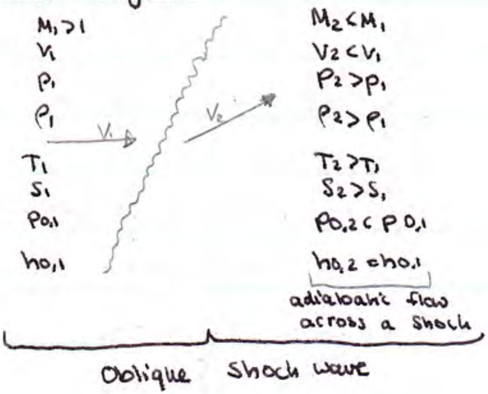
$T_0 = \text{constant}.$

Total pressure and total density are found when the fluid element is not slowed down adiabatically, but isentropically.

Another definition concerns T^*, the temperature when a subsonic fluid element is speeded up to sonic velocity adiabatically, or a hypersonic fluid element is adiabatically slowed down to $M=1$.

p.540-541 In high-speed flows, shock waves occur frequently. The flow properties change drastically over this region.





Thermodynamic relations:

Equation of state: $$p = \rho R T$$ **[7.1]**

For a calorically perfect gas,

$$e = c_v T \quad \text{and} \quad h = c_p T$$ **[7.6a and b]**

$$c_p = \frac{\gamma R}{\gamma - 1}$$ **[7.9]**

$$c_v = \frac{R}{\gamma - 1}$$ **[7.10]**

Forms of the first law:

$$\delta q + \delta w = de$$ **[7.11]**

$$T\,ds = de + p\,dv$$ **[7.18]**

$$T\,ds = dh - v\,dp$$ **[7.20]**

Definition of entropy:

$$ds = \frac{\delta q_{\text{rev}}}{T}$$ **[7.13]**

Also, $$ds = \frac{\delta q}{T} + ds_{\text{irrev}}$$ **[7.14]**

The second law:

$$ds \geq \frac{\delta q}{T}$$ **[7.16]**

or, for an adiabatic process,

$$ds \geq 0$$ **[7.17]**

Entropy changes can be calculated from (for a calorically perfect gas)

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$ **[7.25]**

and $$s_2 - s_1 = c_v \ln \frac{T_2}{T_1} + R \ln \frac{v_2}{v_1}$$ **[7.26]**

For an isentropic flow,

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}$$ **[7.32]**

General definition of compressibility:

$$\tau = -\frac{1}{v}\frac{dv}{dp} \quad \textbf{[7.33]}$$

For an isothermal process,

$$\tau_T = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T \quad \textbf{[7.34]}$$

For an isentropic process,

$$\tau_s = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_s \quad \textbf{[7.35]}$$

The governing equations for inviscid, compressible flow are

Continuity:

$$\frac{\partial}{\partial t}\oiiint_{\mathcal{V}} \rho\, d\mathcal{V} + \oiint_S \rho\mathbf{V}\cdot\mathbf{dS} = 0 \quad \textbf{[7.39]}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\mathbf{V} = 0 \quad \textbf{[7.40]}$$

Momentum:

$$\frac{\partial}{\partial t}\oiiint_{\mathcal{V}} \rho\mathbf{V}\, d\mathcal{V} + \oiint_S (\rho\mathbf{V}\cdot\mathbf{dS})\mathbf{V} = -\oiint_S p\,\mathbf{dS} + \oiiint_{\mathcal{V}} \rho\mathbf{f}\, d\mathcal{V} \quad \textbf{[7.41]}$$

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho f_x \quad \textbf{[7.42a]}$$

$$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho f_y \quad \textbf{[7.42b]}$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho f_z \quad \textbf{[7.42c]}$$

Energy:

$$\frac{\partial}{\partial t}\oiiint_{\mathcal{V}} \rho\left(e + \frac{V^2}{2}\right) d\mathcal{V} + \oiint_S \rho\left(e + \frac{V^2}{2}\right)\mathbf{V}\cdot\mathbf{dS}$$
$$= \oiiint_{\mathcal{V}} \dot{q}\rho\, d\mathcal{V} - \oiint_S p\mathbf{V}\cdot\mathbf{dS} + \oiiint_{\mathcal{V}} \rho(\mathbf{f}\cdot\mathbf{V})\, d\mathcal{V} \quad \textbf{[7.43]}$$

$$\rho\frac{D(e + V^2/2)}{Dt} = \rho\dot{q} - \nabla\cdot p\mathbf{V} + \rho(\mathbf{f}\cdot\mathbf{V}) \quad \textbf{[7.44]}$$

If the flow is steady and adiabatic, Equations (7.43) and (7.44) can be replaced by

$$h_0 = h + \frac{V^2}{2} = \text{const}$$

(*continued*)

Equation of state (perfect gas):

$$p = \rho RT \qquad [7.1]$$

Internal energy (calorically perfect gas):

$$e = c_v T \qquad [7.6a]$$

Total temperature T_0 and total enthalpy h_0 are defined as the properties that would exist if (in our imagination) we slowed the fluid element at a point in the flow to zero velocity adiabatically. Similarly, total pressure p_0 and total density ρ_0 are defined as the properties that would exist if (in our imagination) we slowed the fluid element at a point in the flow to zero velocity isentropically. If a general flow field is adiabatic, h_0 is constant throughout the flow; in contrast, if the flow field is nonadiabatic, h_0 varies from one point to another. Similarly, if a general flow field is isentropic, p_0 and ρ_0 are constant throughout the flow; in contrast, if the flow field is nonisentropic, p_0 and ρ_0 vary from one point to another.

Shock waves are very thin regions in a supersonic flow across which the pressure, density, temperature, and entropy increase; the Mach number, flow velocity, and total pressure decrease; and the total enthalpy stays the same.

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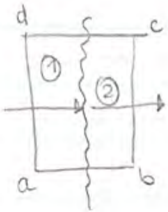
p. 552-554 Some basic normal shock equations can be derived from a control volume around a shock wave. Some observations:



1. Steady flow, $\frac{\partial}{\partial t} = 0$
2. Adiabatic flow, $\dot{q} = 0$ ($T_1 \neq T_2$, though!)
3. No viscous effects on the side of the control volume. However, in the wave, these effects do play a role.
4. No body forces, $\vec{f} = 0$

This yields a simplified form of the continuity equation valid for normal shock waves:

$$\rho_1 u_1 = \rho_2 u_2$$

Similarly, there is a ~~form~~ simpler form of the momentum equation. Again, valid only for normal shocks:

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

As can be expected, the energy equation also has ~~an~~ a form

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that's easier to remember, derived for normal shock waves only:

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

It states that h_0 is constant across the shock wave.

The system of five equations and five unknowns is completed by:

$h_2 = c_p T_2$ (enthalpy)

$p_2 = \rho_2 R T_2$ (equation of state)

p.555-559 Kinetic gas theory states that the molecules of a gas are moving with an average velocity of $\sqrt{8RT/\pi}$. This observation that the velocity only depends on temperature forms the basis in deriving the speed of sound. The flow across a sound wave is adiabatic and ~~isentropic~~ reversible (since the gradients are very small), hence isentropic. From the continuity equation, we find

$$\rho a = (\rho + d\rho)(a + da)$$

$$\rho a = \rho a + a d\rho + \rho da + d\rho da \quad \text{(}d\rho da \rightarrow 0\text{ (product of two differentials))}$$

$$a = -\rho \frac{da}{d\rho}$$

From the momentum equation, we find

$$p + \rho a^2 = (p + dp) + (\rho + d\rho)(a + da)^2$$

$$dp = -2a\rho da - a^2 d\rho$$

$$da = \frac{dp + a^2 d\rho}{2a\rho}$$

Combining these results yields:

$$a = -\rho \frac{dp/d\rho + a^2}{-2a\rho}$$

$$a^2 = \frac{dp}{d\rho}$$

$$\Rightarrow a = \sqrt{\left(\frac{dp}{d\rho}\right)_s} \quad \text{(isentropic process)}$$

The isentropic relations allow for writing this as

$$a = \sqrt{\frac{\gamma p}{\rho}}$$

after which the equation of state provides the final step

$$a = \sqrt{\gamma R T}$$

From the definitions of compressibility and the specific volume, the speed of sound can be written as a function of compressibility:

$$a = \sqrt{\frac{1}{\rho \tau_s}}$$

Evaluating the energy equation provides a final notion: The square of the Mach number is proportional to the ratio of kinetic energy to internal energy in a flow.

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p.564-569 The expression found for the speed of sound introduces new ways of writing the energy equation:

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

$$c_p T_1 + \frac{u_1^2}{2} = c_p T_2 + \frac{u_2^2}{2}$$

$$\frac{a_1^2}{\gamma-1} + \frac{u_1^2}{2} = \frac{a_2^2}{\gamma-1} + \frac{u_2^2}{2}$$

$$\frac{a^2}{\gamma-1} + \frac{u^2}{2} = \frac{a_0^2}{\gamma-1} \quad (a_2 \text{ is a stagnation point})$$

$$\frac{a^2}{\gamma-1} + \frac{u^2}{2} = \frac{\gamma+1}{2(\gamma-1)} a^{*2} \quad (a^* = a_2 \text{ is sonic flow})$$

For these last two equations, it holds that they're constant along a streamline. If the emanate from the same uniform freestream, they're constant throughout the flow.

If a gas is calorically perfect the ratio T/T^* can be expressed as a function of Mach number:

$$\frac{T_0}{T} = 1 + \frac{\gamma-1}{2} M^2$$

Using isentropic relations, pressure- and density ratios are found.

$$\frac{p_0}{p} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\gamma/\gamma-1}$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{1/\gamma-1}$$

For exactly sonic flow, similar equations can be expressed for p^*, ρ^* and T^*:

$$\frac{T^*}{T_0} = \frac{2}{\gamma+1} = 0{,}833 \quad (\gamma = 1{,}4)$$

$$\frac{p^*}{p_0} = \left(\frac{2}{\gamma+1}\right)^{\gamma/\gamma-1} = 0{,}528 \quad (\gamma = 1{,}4)$$

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{1/\gamma-1} = 0{,}634 \quad (\gamma = 1{,}4)$$

Whereas the definition of the Mach number is dealt with previously the characteristic Mach number hasn't been defined:

$$M^* = \frac{V}{a^*}$$

Converting between M and M^* can be done using

$$M^2 = \frac{2}{(\gamma+1)/M^{*2} - (\gamma-1)}$$

$$M^{*2} = \frac{(\gamma+1)M^2}{2 + (\gamma-1)M^2}$$

M^* acts just like M, but for $M \to \infty$, $M^* \to \sqrt{\frac{\gamma+1}{\gamma-1}}$

p.572-575 For $M < 0{,}3$, the flow can be assumed to be incompressible. This assumption is based on the fact that for $M < 0{,}32$, the difference between actual ~~pres~~ density and total density is only 5%. However, in essence, all flows are compressible.

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p.575-581 As stated previously, the 5 unknowns can be solved for using 5 equations; continuity, momentum, energy, the equation of state and the definition of enthalpy. From combinations of these, other expressions can be derived, that sometimes are more convenient.

$$\frac{p_1}{\rho_1 u_1} - \frac{p_2}{\rho_2 u_2} = u_2 - u_1 \quad \text{(from dividing momentum eq. by energy eq.)}$$

$$\frac{a_1^2}{\gamma u_1} - \frac{a_2^2}{\gamma u_2} = u_2 - u_1 \quad \left(a = \sqrt{\gamma \frac{p}{\rho}}\right)$$

$$\left.\begin{aligned} a_1^2 &= \frac{\gamma+1}{2} a^{*2} - \frac{\gamma-1}{2} u_1^2 \\ a_2^2 &= \frac{\gamma+1}{2} a^{*2} - \frac{\gamma-1}{2} u_2^2 \end{aligned}\right\} \text{from energy equation}$$

$$\Rightarrow \frac{\gamma+1}{2}\frac{a^{*2}}{\gamma u_1} - \frac{\gamma-1}{2\gamma} u_1 - \frac{\gamma+1}{2}\frac{a^{*2}}{\gamma u_2} + \frac{\gamma-1}{2\gamma} u_2 = u_2 - u_1$$

$$\frac{\gamma+1}{2\gamma u_1 u_2} a^{*2} + \frac{\gamma-1}{2\gamma} = 1 \quad \text{(dividing by } u_2 - u_1\text{)}$$

$$\Rightarrow a^{*2} = u_1 u_2 \quad \text{(Prandtl relation)}$$

$$1 = M_1^* M_2^* \quad \text{(dividing by } a^{*2}\text{)}$$

$$M_2^* = \frac{1}{M_1^*}$$

$$M_2^2 = \frac{1 + (\gamma-1)/2 \cdot M_1^2}{\gamma M_1^2 - (\gamma-1)/2}$$

This relation, based on the relation between M and M^* and the Prandtl relation, shows two important things:

1. If $M_1 = 1$, then $M_2 = 1$. This happens when an infinitely weak normal shock wave, a Mach wave, occurs.
2. If $M > 1$, then $M_2 < 1$. As $M_1 \to \infty$, $M_2 \to \sqrt{(\gamma-1)/2\gamma}$

Another result of Prandtl's relation is

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{2 + (\gamma-1)M_1^2}$$

The pressure ratio follows from the momentum equation

$$p_2 - p_1 = \rho_1 u_1^2 - \rho_2 u_2^2 = \rho_1 u_1 (u_1 - u_2) = \rho_1 u_1^2 \left(1 - \frac{u_2}{u_1}\right)$$

$$\frac{p_2 - p_1}{p_1} = \frac{\gamma \rho_1 u_1^2}{\gamma p_1}\left(1 - \frac{u_2}{u_1}\right) = \frac{\gamma u_1^2}{a_1^2}\left(1 - \frac{u_2}{u_1}\right) = \gamma M_1^2 \left(1 - \frac{u_2}{u_1}\right)$$

$$\frac{p_2 - p_1}{p_1} = \gamma M_1^2 \left(1 - \frac{2 + (\gamma-1)M_1^2}{(\gamma+1)M_1^2}\right) \quad \text{(from last result above)}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)$$

The equation of state gives the temperature ratio:

$$\frac{T_2}{T_1} = \frac{h_2}{h_1} = \left[1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)\right] \frac{2 + (\gamma-1)M_1^2}{(\gamma+1)M_1^2}$$

From these results, one should note that the Mach number is the ~~im~~ determining parameter for changes across a normal shock wave in a calorically perfect gas.

Mathematically, the above equations hold for any M. However, the second law of thermodynamics dictates they're only valid in nature

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for $M_1 > 1$, hence, for supersonic flow.

When analyzing total pressure and temperature across a shock wave, some conclusions can be drawn.

- $T_{0,1} = T_{0,2}$: the total temperature is constant across a stationary normal shock wave. This makes sense as the flow is adiabatic.
- $\frac{p_{0,2}}{p_{0,1}} = e^{-(s_2 - s_1)/R}$: the total pressure decreases across a shock wave.

p.591 - 593 Whereas a pitot tube can be used to measure velocity from a pressure difference in incompressible flow, it can give the Mach number of a compressible flow. The formulas used, though, depend on the Mach-number regime.

1: Subsonic compressible flow

$$M_1^2 = \frac{2}{\gamma - 1}\left[\left(\frac{p_{0,1}}{p_1}\right)^{(\gamma-1)/\gamma} - 1\right]$$

$$u_1^2 = \frac{2a_1^2}{\gamma - 1}\left[\left(\frac{p_{0,1}}{p_1}\right)^{(\gamma-1)/\gamma} - 1\right]$$

2: supersonic flow

$$\frac{p_{0,2}}{p_1} = \left(\frac{(\gamma+1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma-1)}\right)^{\gamma/\gamma-1} \frac{1-\gamma+2\gamma M_1^2}{\gamma+1}$$ (Rayleigh pitot tube formula)

The speed of sound in a gas is given by

$$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} \qquad \textbf{[8.18]}$$

For a calorically perfect gas,

$$a = \sqrt{\frac{\gamma p}{\rho}} \qquad \textbf{[8.23]}$$

or

$$a = \sqrt{\gamma RT} \qquad \textbf{[8.25]}$$

The speed of sound depends only on the gas temperature.

For a steady, adiabatic, inviscid flow, the energy equation can be expressed as

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \qquad \textbf{[8.29]}$$

$$c_p T_1 + \frac{u_1^2}{2} = c_p T_2 + \frac{u_2^2}{2} \qquad \textbf{[8.30]}$$

$$\frac{a_1^2}{\gamma - 1} + \frac{u_1^2}{2} = \frac{a_2^2}{\gamma - 1} + \frac{u_2^2}{2} \qquad \textbf{[8.32]}$$

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{a_0^2}{\gamma - 1} \qquad \textbf{[8.33]}$$

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{\gamma + 1}{2(\gamma - 1)} a^{*2} \qquad \textbf{[8.35]}$$

Total conditions in a flow are related to static conditions via

$$c_p T + \frac{u^2}{2} = c_p T_0 \qquad \textbf{[8.38]}$$

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2 \qquad \textbf{[8.40]}$$

$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma - 1)} \qquad \textbf{[8.42]}$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{1/(\gamma - 1)} \qquad \textbf{[8.43]}$$

Note that the ratios of total to static properties are a function of local Mach number only. These functions are tabulated in Appendix A.

The basic normal shock equations are

Continuity: $$\rho_1 u_1 = \rho_2 u_2 \quad \textbf{[8.2]}$$

Momentum: $$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \quad \textbf{[8.6]}$$

Energy: $$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \quad \textbf{[8.10]}$$

These equations lead to relations for changes across a normal shock as a function of upstream Mach number M_1 only:

$$M_2^2 = \frac{1 + [(\gamma - 1)/2]M_1^2}{\gamma M_1^2 - (\gamma - 1)/2} \quad \textbf{[8.59]}$$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2} \quad \textbf{[8.61]}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1) \quad \textbf{[8.65]}$$

$$\frac{T_2}{T_1} = \frac{h_2}{h_1} = \left[1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)\right]\frac{2 + (\gamma - 1)M_1^2}{(\gamma + 1)M_1^2} \quad \textbf{[8.67]}$$

$$s_2 - s_1 = c_p \ln\left\{\left[1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)\right]\frac{2 + (\gamma - 1)M_1^2}{(\gamma + 1)M_1^2}\right\} - R\ln\left[1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)\right] \quad \textbf{[8.68]}$$

$$\frac{p_{0,2}}{p_{0,1}} = e^{-(s_2 - s_1)/R} \quad \textbf{[8.73]}$$

The normal shock properties are tabulated versus M_1 in Appendix B.

For a calorically perfect gas, the total temperature is constant across a normal shock wave:

$$T_{0,2} = T_{0,1}$$

However, there is a loss in total pressure across the wave:

$$p_{0,2} < p_{0,1}$$

For subsonic and supersonic compressible flow, the freestream Mach number is determined by the ratio of Pitot pressure to freestream static pressure. However, the equations are different:

Subsonic flow:

$$M_1^2 = \frac{2}{\gamma - 1}\left[\left(\frac{p_{0,1}}{p_1}\right)^{(\gamma-1)/\gamma} - 1\right] \quad \textbf{[8.74]}$$

Supersonic flow:

$$\frac{p_{0,2}}{p_1} = \left[\frac{(\gamma+1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma - 1)}\right]^{\gamma/(\gamma-1)} \frac{1 - \gamma + 2\gamma M_1^2}{\gamma + 1} \quad \textbf{[8.80]}$$

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p. 602-606 Normal shock waves provided a relatively easy introduction. However, most supersonic flows are characterized by oblique shock waves and oblique expansion waves, where the pressure decreases continuously along the wave. If a supersonic flow meets a concave corner and "turns into itself", an oblique shock occurs. Convex corners make the flow "turn away from itself" and result in oblique expansion waves. Across this expansion wave, the Mach number increases and pressure, density and temperature decrease, making the expansion wave the antithesis of a shock wave.

The angle of a Mach wave is a function of Mach number only:

$$\mu = \sin^{-1}\left(\frac{1}{M}\right)$$

The angle an oblique shock waves makes with the free-stream is bigger than the Mach angle:

$$\beta > \mu$$

However, the Mach wave (with associated angle μ) is the limiting case of a shock wave.

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p.608-610 The wave angle β just defined is the angle between the shock wave and the upstream flow direction (Fig. 9.8, p. 609).
When it hits the shock wave, the velocity vector can be split up in a normal and a tangential component.
Evaluating the continuity equation over a control volume across a shock wave, and noting the fact that the flow is steady, inviscid and adiabatic, a simplified expression is found:

$$\rho_1 u_1 = \rho_2 u_2 \quad (u_1 \text{ and } u_2 \text{ perpendicular to shock})$$

The momentum equation can also be simplified. Resolving the velocity into a tangential and a normal contribution yields

$$\oint\!\!\oint_S (\rho \vec{V} \cdot \vec{dS}) w = - \oint\!\!\oint_S (p\, dS)_{\text{tang / norm}}$$

$$\Rightarrow -(\rho_1 u_1 A_1) w_1 + (\rho_2 u_2 A_2) w_2 \quad (\text{tangential direction})$$

$$w_1 = w_2 \quad (\text{tangential direction})$$

$$\Rightarrow -(\rho_1 u_1 A_1) u_1 + (\rho_2 u_2 A_2) u_2 = -(-p_1 A_1 + p_2 A_2) \quad (\text{normal direction})$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \quad (\text{normal direction})$$

Finally, the energy equation is considered.

$$\oint\!\!\oint_S \rho \left(e + \frac{V^2}{2}\right) \vec{V} \cdot \vec{dS} = - \oint\!\!\oint_S p\vec{V} \cdot \vec{dS}$$

$$-\rho_1 \left(e_1 + \frac{V_1^2}{2}\right) u_1 A_1 + \rho_2 \left(e_2 + \frac{V_2^2}{2}\right) u_2 A_2 = -(-p_1 u_1 A_1 + p_2 u_2 A_2)$$

$$\rho_1 u_1 \left(h_1 + \frac{V_1^2}{2}\right) = \rho_2 u_2 \left(h_2 + \frac{V_2^2}{2}\right) \quad \text{division by continuity equation}$$

$$\Rightarrow h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$$

"total enthalpy is constant across the shock wave"
"total temperature is constant across a shock wave in a calorically perfect gas"

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

As these results only depend on u_1 and u_2, the normal velocity components, we can conclude that changes across an oblique shock wave are governed only by the component of velocity normal to the wave. Also note that they're equal to the governing equations of a normal shock wave, derived earlier. So, the results obtained earlier (and repeated below) are also valid for the normal component over oblique shock waves. This is given by simple geometry:

$$M_n = M \sin \beta$$

$$M_{n,2}^2 = \frac{1+(\gamma-1)/2 \cdot M_{n,1}^2}{\gamma M_{n,1}^2 - (\gamma-1)/2}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_{n,1}^2}{2+(\gamma-1) M_{n,1}^2}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_{n,1}^2 - 1)$$

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} \quad \text{(from Equation of State)}$$

The Mach number behind the shock wave can be computed using:

$$M_2 = \frac{M_{n,2}}{\sin(\beta-\theta)}$$

θ is the so-called deflection angle (fig. 9.8, p. 609). It is a function of M_1 and β:

$$\left.\begin{array}{l} \tan\beta = \frac{u_1}{w_1} \\ \tan(\beta-\theta) = \frac{u_2}{w_2} \end{array}\right\} \Rightarrow \frac{\tan(\beta-\theta)}{\tan(\beta)} = \frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} = \frac{2+(\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)M_1^2\sin^2\beta}$$

$$\Rightarrow \tan\theta = 2\cot(\beta)\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma+\cos(2\beta))+2} \quad (\theta\text{-}\beta\text{-}M\text{ - relation})$$

When this relation is plotted (fig. 9.9, p. 613) it shows a wealth of phenomena associated with oblique shock waves.

1. For any M_1, there is a maximum deflection angle θ_{max}. If $\theta > \theta_{max}$ because of geometry, no straight shock exists, but a curved one will show. This shock wave is detached from the nose of the body.
   Since θ_{max} increases with increasing M_1 the straight shock can exist at higher θ for higher M_1. In the limit of $M \to \infty$, we have $\theta_{max} \to 44.5°$ (for $\gamma = 1.4$; hence, air)
2. For $\theta < \theta_{max}$, two solutions exist for a given M_1. The smaller angle corresponds to the weak shock solution, the larger to the strong. This strong wave will show a larger pressure ratio $\frac{p_2}{p_1}$. In nature, the weak wave usually prevails.
   If the strong shock occurs, $M_2 < 1$. This also happens for weak waves at θ close to θ_{max}. For the weak shock solution, $M_2 > 1$.
3. If $\theta = 0$, $\beta = 90°$ (normal shock) or $\beta = \mu$. In both cases, the flow streamlines experience no deflection across the wave.
4. In general, for attached shocks with a fixed (by geometry) θ, β decreases with increasing M_1 and the shock wave becomes stronger. This is because the increase of M_1 has a larger effect on $M_{n,1}$ than the decrease of β.

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5. In general, for attached shocks with a fixed upstream Mach number, β increases with increasing θ and the shock wave becomes stronger.

p.622-626 The 2D-theory developed provides exact solutions for the flow over 2D bodies. If these are 3D (cone vs wedge, for example), things change. This is because of the "three-dimensional relieving effect" (Chapter 6); ~~the~~ the flow has one more dimension to get out of the way of the body.

- the 2D wave angle is larger than the 3D wave angle;
- the deflection angle is smaller in the 3D case, but since the body angle (geometry) is equal, it has to gradually curve;
- the pressure on the 3D cone is less than on the wedge surface;
- the Mach number on the cone is larger than on the 2D body.

Or, stated differently:

- 3D shock wave is weaker
- 3D surface pressure is less
- Streamlines around the 3D body are curved rather than straight.

Aerodynamic coefficients only depend on the Mach number as long as the flow is inviscid.

p.628-631 When a shock wave impinges on a solid boundary, a reflected shock wave forms. Physically, it's a mechanism to preserve the flow-tangency condition by "cancelling" the deflection angle. Because the reflected wave is weaker than the incident wave, the wave reflection is not specular: $\Phi \neq \beta_1$. The properties in the region behind the reflected shock (3) can be computed as follows.

1. Calculate the properties in the region behind the main, but before the reflected wave (region 2). ~~For~~ This gives M_2 from M_1 and θ.
2. Calculate the properties in region 3, from M_2 and θ.

Some special situations can arise:

- Mach reflection: when M_1 is only slightly larger than 1, the shock wave turns perpendicular to the boundary. The "reflection"

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travels back and then branches to continue downstream.

- Intersection of a left- and a right-running wave: after intersecting, the ~~two~~ shocks are refracted. The region between these two refracted waves is divided by a slip line, along which the pressures and the direction of the velocity (parallel to the slip line), but not necessarily the magnitudes, are constant.
- Intersection of two left (or right) running waves: after intersection, the waves merge as a stronger one, along with a weak reflection. This situation also has a slip line, starting in the intersection.

Of course, there are way more possibilities. These three, however, are most common.

p.632-635 δ is the symbol used for shock detachment distance, and indicates the distance between a bow shock and the nose of a blunt body. Such a bow shock is one of the instances in nature when you can observe all possible oblique shock solutions at once for a given M_1. Regions of both subsonic and supersonic flow occur (subsonic near the nose and corresponding normal shock, supersonic further away). These are divided by a so-called sonic line.

The streamline wetting the body is the streamline with the largest entropy, since it passes through the strongest part (normal) of the shock wave. Streamlines further away are at lower s, such that a entropy gradient can be identified. This gradient induces vorticity, which relation is quantified by Crocco's theorem:

$$T \nabla s = \nabla h_0 - \vec{V} \times (\nabla \times \vec{V})$$

($T\nabla s$ ↳ entropy gradient)

From this, we can conclude that the flow field behind a curved shock is rotational.

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p.636-641 As discussed before, closely related to shock waves are expansion waves. These occur when a supersonic flow is turned away from itself. When expansion waves form, "expansion fans" can be seen. These are bounded by two angles:

$$\mu_1 = \sin^{-1}\left(\frac{1}{M_1}\right) \quad \text{(forward Mach line)}$$
$$\mu_2 = \sin^{-1}\left(\frac{1}{M_2}\right) \quad \text{(rearward Mach line)}$$

The expansion is isentropic, which is in contrast to the case of an oblique shock.

Prandtl and Meyer worked out a theory for ~~[crossed out]~~ centered expansion waves (emanating from a sharp convex corner). Therefore, these are also known as Prandtl-Meyer expansion waves. It focuses on finding the flow properties in the region behind the expansion fan (i.e., behind the rearward Mach line) based on knowing the properties of the freestream and the (geometric) deflection angle θ.

Modeling the expansion wave as ~~an~~ a Mach wave inclined μ and with $\theta = d\theta \to 0$, geometry allows for finding the relation between an infinitesimal velocity increase to an incremental change in deflection

$$d\theta = \sqrt{M^2-1}\,\frac{dV}{V}$$

Integrating from region 1 (before) ~~[crossed out]~~ to 2 (after) and substituting a ~~[crossed out]~~ bunch of other equations yields the Prandtl-Meyer function:

$$\int_0^{\theta} d\theta = \int_{M_1}^{M_2} \sqrt{M^2-1}\,\frac{dV}{V}$$

$$M = \frac{V}{a} \Rightarrow V = M \cdot a$$

$$\ln(V) = \ln(M)\cdot\ln(a)$$
$$\frac{dV}{V} = \frac{dM}{M}\cdot\frac{da}{a}$$

$$a = \sqrt{\gamma R T}, \quad \frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2 \;\Big\}\; \frac{a_0^2}{a} = \frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2$$

$$a = a_0\left(1+\frac{\gamma-1}{2}M^2\right)^{-\frac{1}{2}}$$

$$\frac{da}{a} = -\left(\frac{\gamma-1}{2}\right)M\left(1+\frac{\gamma-1}{2}M^2\right)^{-1}dM$$

$$\Rightarrow \frac{dV}{V} = \frac{1}{1+(\gamma-1)/2\cdot M^2}\,\frac{dM}{M}$$

$$\theta = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+(\gamma-1)/2\cdot M^2}\,\frac{dM}{M} = \nu(M)$$

↳ Prandtle-Meyer function

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left(\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)}\right) - \tan^{-1}\left(\sqrt{M^2-1}\right) \quad \text{(calorically perfect gas)}$$

Setting the constant of integration to zero (such that ~~[crossed out]~~ $= \nu(1) = 0$)

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allows for writing θ as a function of M.

$$\theta = \nu(M_2) - \nu(M_1)$$

This function is an important part of the problem-solving strategy for expansion waves:

1. ~~Obtain~~ Obtain $\nu(M_1)$ from Appendix C.
2. Calculate $\nu(M_2)$ with known $\nu(M_1)$ and θ
3. Obtain M_2 from $\nu(M_2)$ from, again, App. C.
4. Find pressure and temperature ~~re~~ ratios with isentropic relations:

$$\frac{T_2}{T_1} = \frac{T_2/T_{0,2}}{T_1/T_{0,1}} = \frac{1+(\gamma-1)/2 \cdot M_1^2}{1+(\gamma-1)/2 \cdot M_2^2}$$

$$\frac{p_2}{p_1} = \frac{p_2/p_0}{p_1/p_0} = \left(\frac{1+(\gamma-1)/2 \cdot M_1^2}{1+(\gamma-1)/2 \cdot M_2^2}\right)^{\gamma/\gamma-1}$$

p.648-651 Considering a flat plate under an angle of attack in a supersonic flow, one can identify expansion waves at the top leading edge and the bottom trailing edge and shocks at the top trailing edge and bottom leading edge. Both surfaces experience a uniform pressure distribution, with p_3 (bottom) > p_2 (top), ~~exerti~~ resulting in an aerodynamic force R' (per unit span, chord c):

$$R' = (p_3 - p_2)c$$

$$L' = (p_3 - p_2)c \cdot \cos\alpha$$

$$D' = (p_3 - p_2)c \cdot \sin\alpha$$

Shock-expansion theory states that when a body consists ~~of a series~~ of straight-line segments with deflection angles small enough to ensure no detached shock waves are formed, the flow goes through a series of distinct oblique shock and expansion waves. ~~From this~~ Because of this, the theories developed earlier hold exactly.

An important aspect of inviscid supersonic flow is wave drag. It is related to the increasing entropy and (hence) the loss of total pressure across the oblique shock wave. d'Alembert ~~principle~~ paradox doesn't hold for supersonic flow.

P.652 To find ~~~~ lift- and drag coefficients, we only need to know the shape of the body, the angle of attack and the free-stream Mach number.

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p.657-658 Shock waves and boundary layers do not mix, but interactions frequently occur. When an incident wave impinges, it creates an adverse pressure gradient that is infinitely large. This causes the boundary layer to separate from the surface, and turns the flow into itself. This, in turn, leads to a second wave. The separated boundary layer subsequently turns back towards the plate and reattaches. The flow is again turned into itself, resulting in a third, so-called "re-attachment shock". Between the separation and reattachment waves, the flow is turned away from itself and expansion waves are formed. Where the boundary layer reattaches, it's relatively thin, resulting in high temperatures.
Each boundary layer - shock wave interaction yields the behavior described above, but the severity of the effects depend on whether the boundary layer is laminar (easier separation) or turbulent.

An infinitesimal disturbance in a multidimensional supersonic flow creates a Mach wave which makes an angle μ with respect to the upstream velocity. This angle is defined as the Mach angle and is given by

$$\mu = \sin^{-1} \frac{1}{M} \tag{9.1}$$

Changes across an oblique shock wave are determined by the normal component of velocity ahead of the wave. For a calorically perfect gas, the normal component of the upstream Mach number is the determining factor. Changes across an oblique shock can be determined from the normal shock relations derived in Chapter 8 by using $M_{n,1}$ in these relations, where

$$M_{n,1} = M_1 \sin \beta \tag{9.13}$$

Changes across an oblique shock depend on two parameters, for example, M_1 and β, or M_1 and θ. The relationship between M_1, β, and θ is given in Figure 9.7, which should be studied closely.

Oblique shock waves incident on a solid surface reflect from that surface in such a fashion to maintain flow tangency on the surface. Oblique shocks also intersect each other, with the results of the intersection depending on the arrangement of the shocks.

The governing factor in the analysis of a centered expansion wave is the Prandtl-Meyer function $\nu(M)$. The key equation which relates the downstream Mach number M_2, the upstream Mach number M_1, and the deflection angle θ is

$$\theta = \nu(M_2) - \nu(M_1) \tag{9.43}$$

The pressure distribution over a supersonic airfoil made up of straight-line segments can usually be calculated exactly from a combination of oblique and expansion waves—that is, from exact shock-expansion theory.

p672-681 One-dimensional flow is, strictly speaking constant-area flow. If the area varies as a function of x, the flow ~~can~~ becomes three-dimensional. However since the y- and z-velocities are small, it can be assumed to still be one-dimensional or, better, quasi-one-dimensional flow.

It is governed by forms of well-known equations. From the continuity equation, we find

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

Whereas the momentum equation simplifies to

$$\frac{\partial}{\partial t}\iiint_V \rho\vec{V}\cdot dV + \oiint_S (\rho\vec{V}\cdot d\vec{S})\vec{V} = -\oiint_S p\,d\vec{S} + \iiint_V \rho\vec{f}\,dV + \vec{F}_{visc}$$

$$\oiint_S (\rho\vec{V}\cdot d\vec{S})\vec{V} = -\oiint_S p\,d\vec{S} \quad \text{(inviscid, no body forces, steady)}$$

$$\oiint_S (\rho\vec{V}\cdot d\vec{S})u = -\oiint_S (p\,d\vec{S})_x$$

$$-\rho_1 u_1^2 A_1 + \rho_2 u_2^2 A_2 = -(-p_1 A_1 + p_2 A_2) + \int_{A_1}^{A_2} p\,dA$$

(dA ↳ x-component of $d\vec{S}$)

$$p_1 A_1 + \rho_1 u_1^2 A_1 + \int_{A_1}^{A_2} p\,dA = p_2 A_2 + \rho_2 u_2^2 A_2 \quad \text{(steady, quasi-one-dimensional)}$$

Finally the energy equation yields:

Eq. 2.95

$$\oiint_S \rho\left(e + \frac{V^2}{2}\right)\vec{V}\,d\vec{S} = -\oiint_S p\vec{V}\cdot d\vec{S} \quad \text{(inviscid, adiabatic, steady, no body forces)}$$

$$\rho_1\left(e_1 + \frac{u_1^2}{2}\right)(-u_1 A_1) + \rho_2\left(e_2 + \frac{u_2^2}{2}\right)(u_2 A_2) = -(-p_1 u_1 A_1 + p_2 u_2 A_2)$$

$$p_1 u_1 A_1 + \rho_1 u_1 A_1\left(e_1 + \frac{u_1^2}{2}\right) = p_2 u_2 A_2 + \rho_2 u_2 A_2\left(e_2 + \frac{u_2^2}{2}\right)$$

$$\frac{p_1}{\rho_1} + e_1 + \frac{u_1^2}{2} = \frac{p_2}{\rho_2} + e_2 + \frac{u_2^2}{2}$$ (division by continuity equation)

$h = e + pv = e + p/\rho$

$$\Rightarrow h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$ (steady, adiabatic, inviscid, quasi-one-dimensional)

h_0 = constant

These three equations combined with the equation of state and the second law of thermodynamics allow for solving the five state variables. Using differential equations however, is easier.

$d(\rho u A) = 0$ (continuity equation)

$p_1 = p$; $\rho_1 = \rho$; $u_1 = u$

$p_2 = p + dp$; $\rho_2 = \rho + d\rho$; $u_2 = u + du$

$$\Rightarrow pA + \rho u^2 A + p\,dA = (p+dp)(A+dA) + (\rho + d\rho)(u+du)^2(A+dA)$$

$$A\,dp + Au^2 d\rho + \rho u^2 dA + 2\rho u A\,du = 0 \quad (dx \cdot dy \approx 0)$$

$$\rho u^2 dA + \rho u A\,du + Au^2 d\rho = 0$$

$$dp = -\rho u\,du$$ (momentum equation) / Euler's equation

$dh + u\,du = 0$ (energy equation)

From these differential equations, we can find some physical characteristics of quasi-1D flow.

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0 \quad ①$$

$$\frac{dp}{\rho} = \frac{dp}{d\rho}\frac{d\rho}{\rho} = -u\,du$$

$$\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s = a^2$$ (inviscid, adiabatic, isentropic)

$$a^2 \frac{d\rho}{\rho} = -u\,du$$

$$\frac{d\rho}{\rho} = -\frac{u\,du}{a^2} = -\frac{u^2}{a^2}\frac{du}{u} = -M^2\frac{du}{u} \quad ②$$

$$-M^2\frac{du}{u} + \frac{du}{u} + \frac{dA}{A} = 0 \quad (② \to ①)$$

$$\frac{dA}{A} = (M^2 - 1)\frac{du}{u}$$ (area-velocity relation)

This equation tells us the following information:

1. $0 \leq M \leq 1$ (subsonic flow): an increase in velocity corresponds to a decrease in area and vice versa
2. $M > 1$ (supersonic flow): an increase in velocity is associated with an increase in area, and the other way around.
3. $M = 1$ (sonic flow). Sonic flow occurs at the minimum area, known by the throat.
4. $M = 0$: this implies Au = constant, which also follows from the continuity equation.

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p.681-680 As introduced before, an asterisk identifies sonic conditions. Hence, in the throat of a supersonic wind tunnel, $M^* = 1$ and $a^* = u^*$. This allows for rewriting the continuity equation:

$$\underbrace{\rho^* u^* A^*}_{\text{throat}} = \underbrace{\rho u A}_{\text{elsewhere}}$$

From this, an equation known as the area-Mach number relation can be derived:

$$\left.\begin{array}{l}\rho^* u^* A^* = \rho u A \\ u^* = a^*\end{array}\right\} \quad \frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{a^*}{u} = \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\frac{a^*}{u} \quad (1)$$

ρ_0 ↳ stagnation density, constant through isentropic flow

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)} \quad (2)$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{1/(\gamma-1)} \quad (3)$$

$$\left(\frac{u}{a^*}\right)^2 = M^{*2} = \frac{(\gamma+1)/2 \cdot M^2}{1 + (\gamma-1)/2 \cdot M^2} \quad (4)$$

(2) – (4) → (1)

$$\left(\frac{A}{A^*}\right)^2 = \left(\frac{\rho^*}{\rho_0}\right)^2 \left(\frac{\rho_0}{\rho}\right)^2 \left(\frac{a^*}{u}\right)^2$$

$$\left(\frac{A}{A^*}\right)^2 = \left(\frac{2}{\gamma+1}\right)^{2/\gamma-1} \cdot \left(1 + \frac{\gamma-1}{2}M^2\right)^{2/\gamma-1} \frac{1 + (\gamma-1)/2 \cdot M^2}{(\gamma+1)/2 \cdot M^2}$$

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2}M^2\right)\right]^{\gamma+1/\gamma-1} \quad \text{(area-Mach number relation)}$$

It tells that the Mach number only is a function of the ratio of local to throat areas. The equation above yields two possible M, noting that $\frac{A}{A^*} \geq 1$ (isentropic flow). Which of these two holds depends on inlet and exit pressures. Knowing the ratio of area's yields the Mach distribution, which gives pressure and temperature distributions.

When the ratio between inlet and exit pressures is decreased, the throat velocity increases to a state of sonic flow. When the exit pressure is now reduced even further, M will remain constant: ~~from~~ in the throat, the Mach number cannot be larger than 1. ~~[struck-out text]~~ Consequently, the mass flow, $\dot{m} = \rho_t u_t A_t = \rho^* u^* A_t$, reduces. This situation, sonic flow at the throat and constant mass flow with decreasing exit pressure is ~~chocked~~ choked flow.

Although nothing happens before and in the throat, a lot of things occur in the diverging section. Most notably, a region of supersonic flow appears when the exit pressure is reduced below $p_{e,3}$, the

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value corresponding to ~~choked~~ choked flow. The supersonic flow doesn't stretch to the exit, however, and a normal shock wave is formed. When $p_e = p_{e,4}$, the shock is formed at a distance d from the throat. Before the shock, the supersonic isentropic solution holds, ~~and~~ ~~results~~ behind, it slows down isentropically. Across the shock, there is a pressure discontinuity (increasing), and a discontinuous increase in entropy.

The ~~to~~ value of d ~~depends~~ is given by the requirement that the increase in static pressure across the wave plus that in the divergent portion of the subsonic flow behind the shock be just right to achieve $p_{e,4}$ at the exit. For $p_{e,5}$, the shock moves to the exit precisely. Then, the flow between the throat and the exit (but not at the exit itself) is isentropic.

Instead of stating p_e is changed, it can be said that the back pressure p_B, the pressure of the air of the surroundings ~~behind the~~ downstream of the exit, is changed. This is true because $p_e = p_B$, since a pressure discontinuity cannot exist in a steady subsonic flow. (This holds when the flow at the exit is subsonic)

When p_B is reduced below $p_{e,5}$, but kept above $p_{e,6}$ (which corresponds to isentropic pressure). To make this happen, the gas has to be compressed. This takes place ~~through~~ across an oblique shock. For $p_B = p_{e,6}$, there is no "pressure mismatch". For $p_B < p_{e,6}$, expansion waves occur to "bridge the gap". This state is called underexpanded, whereas $p_B > p_{e,6}$ is identified as overexpanded.

p. 696-698 Diffusers are designed to slow down a gas. The shape drastically depends on whether the flow is sub- or supersonic. A diffuser should perform its task with as small a loss in total pressure as possible. Hence, an ideal diffuser compresses isentropically. Unfortunately, this is not possible, and entropy increases between inlet and exit.

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p. 699-703 There are a number of ways to achieve a certain pressure ratio across a nozzle in a supersonic wind tunnel. It's possible to have $p_0 = p_e$, but then p_0 has to be large. A more efficient way is to add a constant-area section to the nozzle. A normal shock will occur at its end, ~~reducing~~ effectively reducing the exit pressure, resulting in a much lower inlet pressure. This normal shock acts as a diffuser. Using a normal shock as a diffuser has some disadvantages, however:

1. Normal shocks result in a large total pressure loss;
2. Fixing a normal shock at the exit is (almost) impossible due to flow unsteadyness and instabilities;
3. A test object, ~~introducing oblique shocks, that~~ will introduce oblique shocks that make the flow 3 dimensional, in which a normal shock cannot exist.

The idea of having a diffuser is a good one, though, and an arrangement of a convergent-divergent nozzle; a (constant area) test section and a convergent-divergent diffuser ~~is the to~~ comprises a basic supersonic wind tunnel. Most of the time, this set-up is more efficient than a simple normal shock.

This set-up has two throats: a nozzle throat (1) and a diffuser throat (2). As the mass flow remains constant,

$$\rho_1^* a_1^* A_{t,1} = \rho_2 u_2 A_{t,2}$$

and because of irreversibility across the shock waves, ρ_2 and u_2 are (most likely) not equal to ρ_1^* and a_1^*. Hence:

$$\frac{A_{t,2}}{A_{t,1}} = \frac{\rho_1^* a_1^*}{\rho_2 a_2^*} \quad \text{(assuming sonic flow in both throats)}$$

$$\frac{A_{t,2}}{A_{t,1}} = \frac{\rho_1^*}{\rho_2^*} \quad \text{(adiabatic flow across shocks, for which } a_1^* = a_2^*\text{)}$$

$$\frac{A_{t,2}}{A_{t,1}} = \frac{p_1^*}{p_2^*} \quad \text{(equation of state, T constant for adiabatic flow)}$$

$$p = p_0 \left(\frac{2}{\gamma+1}\right)^{\gamma/\gamma-1}$$

$$\Rightarrow \frac{A_{t,2}}{A_{t,1}} = \frac{p_{0,1}}{p_{0,2}}$$

Since the total pressure decreases across a shock wave, $p_{0,2} < p_{0,1}$, and $A_{t,2}$ (diffuser) > $A_{t,1}$ (nozzle). If the diffuser area is too small, the diffuser will choke, resulting in extra shocks in the tunnel, ~~and~~ a reduced Mach number in the test section and an overall reduction

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in total pressure loss. In this case, the wind tunnel is said to be unstarted. Adjusting the area-ratio can solve this problem.

Quasi-one-dimensional flow is an approximation to the actual three-dimensional flow in a variable-area duct; this approximation assumes that $p = p(x)$, $u = u(x)$, $T = T(x)$, etc., although the area varies as $A = A(x)$. Thus, we can visualize the quasi-one-dimensional results as giving the mean properties at a given station, averaged over the cross section. The quasi-one-dimensional flow assumption gives reasonable results for many internal flow problems; it is a "workhorse" in the everyday application of compressible flow. The governing equations for this are

Continuity:
$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2 \qquad \textbf{[10.1]}$$

Momentum:
$$p_1 A_1 + \rho_1 u_1^2 A_1 + \int_{A_1}^{A_2} p\, dA = p_2 A_2 + \rho_2 u_2^2 A_2 \qquad \textbf{[10.5]}$$

Energy:
$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \qquad \textbf{[10.9]}$$

The area velocity relation

$$\frac{dA}{A} = (M^2 - 1)\frac{du}{u} \qquad \textbf{[10.25]}$$

tells us that

1. To accelerate (decelerate) a subsonic flow, the area must decrease (increase).
2. To accelerate (decelerate) a supersonic flow, the area must increase (decrease).
3. Sonic flow can only occur at a throat or minimum area of the flow.

The isentropic flow of a calorically perfect gas through a nozzle is governed by the relation

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{(\gamma+1)/(\gamma-1)} \qquad \textbf{[10.32]}$$

This tells us that the Mach number in a duct is governed by the ratio of local duct area to the sonic throat area; moreover, for a given area ratio, there are two values of Mach number that satisfy Equation (10.32)—a subsonic value and a supersonic value.

For a given convergent-divergent duct, there is only one possible isentropic flow solution for supersonic flow; in contrast, there are an infinite number of subsonic isentropic solutions, each one associated with a different pressure ratio across the nozzle, $p_0/p_e = p_0/p_B$.

In a supersonic wind tunnel, the ratio of second throat area to first throat area should be approximately

$$\frac{A_{t,2}}{A_{t,1}} = \frac{p_{0,1}}{p_{0,2}} \qquad \textbf{[10.38]}$$

If $A_{t,2}$ is reduced much below this value, the diffuser will choke and the tunnel will unstart.

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p.713 When analyzing ~~super~~ compressible flow over airfoils, it is necessary to go back to partial differential forms of the continuity, momentum and energy equations. These are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \quad \text{Continuity equation}$$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho f_x + (F_x)_{viscous}$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho f_y + (F_y)_{viscous}$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho f_z + (F_z)_{viscous}$$

momentum equation

$$\rho \frac{D(e + V^2/2)}{Dt} = \rho \dot{q} - \nabla \cdot (p\vec{V}) + \rho (\vec{f} \cdot \vec{V}) + \dot{Q}'_{viscous} + \dot{W}'_{viscous}$$

energy equation

p.714-716 These governing equations can be combined into one, depending on the velocity potential, analogous to ~~the~~ Laplace's equation (for incompressible). We can do this because the flow is irrotational.

$$\vec{V} = \nabla \varphi \quad \text{(2D, steady, irrotational and inviscid flow)}$$

$$u = \frac{\partial \varphi}{\partial x}$$

$$v = \frac{\partial \varphi}{\partial y}$$

$$\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + \rho \frac{\partial v}{\partial y} = 0 \quad \text{(continuity equation)}$$

$$\Rightarrow \rho \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right) + \frac{\partial \varphi}{\partial x}\frac{\partial \rho}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{\partial \rho}{\partial y} = 0$$

$$dp = -\rho V \, dV \quad \text{(momentum equation)}$$

$$dp = -\rho \cdot \tfrac{1}{2} d(V^2) = -\frac{\rho}{2} d(u^2 + v^2)$$

$$= -\frac{\rho}{2} d\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 \right]$$

$$\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s \quad \text{(isentropic flow)}$$

$$\left(\frac{\partial p}{\partial \rho}\right)_s = a^2$$

$$dp = a^2 d\rho$$

$$\Rightarrow d\rho = -\frac{\rho}{2a^2} d\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 \right]$$

$$\frac{\partial \rho}{\partial x} = -\frac{\rho}{2a^2} \frac{\partial}{\partial x}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 \right] = -\frac{\rho}{a^2}\left(\frac{\partial \varphi}{\partial x}\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial \varphi}{\partial y}\frac{\partial^2 \varphi}{\partial x \partial y} \right)$$

$$\frac{\partial \rho}{\partial y} = -\frac{\rho}{2a^2} \frac{\partial}{\partial y}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 \right] = -\frac{\rho}{a^2}\left(\frac{\partial \varphi}{\partial x}\frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial \varphi}{\partial y}\frac{\partial^2 \varphi}{\partial y^2} \right)$$

$$\Rightarrow \left[1 - \frac{1}{a^2}\left(\frac{\partial \varphi}{\partial x}\right)^2\right] \frac{\partial^2 \varphi}{\partial x^2} + \left[1 - \frac{1}{a^2}\left(\frac{\partial \varphi}{\partial y}\right)^2\right] \frac{\partial^2 \varphi}{\partial y^2} - \frac{2}{a^2}\left(\frac{\partial \varphi}{\partial x}\right)\left(\frac{\partial \varphi}{\partial y}\right)\frac{\partial^2 \varphi}{\partial x \partial y} = 0$$

(Velocity potential equation)

$$a^2 = a_0^2 - \frac{\gamma - 1}{2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 \right]$$

Hence, this equation gives an expression for the velocity potential and the speed of sound. However, that can also be expressed in terms of φ. With this velocity potential equation, the number of equations that has to be solved reduces to only one, and all other ~~system~~ variables can be computed using the following methods:

1. u and v from $u = \frac{\partial \varphi}{\partial x}$ and $v = \frac{\partial \varphi}{\partial y}$
2. a from $a^2 = a_0^2 - \frac{\gamma - 1}{2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right]$
3. M from $M = V/a = \sqrt{u^2 + v^2}/a$
4. T, p and ρ from Mach relations

~~The~~ An important disadvantage of the velocity potential equation is the fact that it is non-linear (unlike Laplace's equation). Hence, solving it almost always requires numerical techniques.

p.717-722 However, it is possible to solve a linearized approximation of the velocity potential equation. In its derivation, perturbation velocities are ~~define~~ used, defined as $\hat{u}$ and $\hat{v}$, such that $u = V_\infty + \hat{u}$ and $v = \hat{v}$. This also allows for setting up a perturbation velocity potential equation:

$$\varphi = V_\infty x + \hat{\varphi}$$

$\hat{\varphi}$ → perturbation velocity potential

$$\frac{\partial \hat{\varphi}}{\partial x} = \hat{u} \qquad \frac{\partial \hat{\varphi}}{\partial y} = \hat{v}$$

$$\Rightarrow \frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \hat{\varphi}}{\partial x^2} \qquad \frac{\partial^2 \varphi}{\partial y^2} = \frac{\partial^2 \hat{\varphi}}{\partial y^2} \qquad \frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial^2 \hat{\varphi}}{\partial x \partial y}$$

$$\Rightarrow \left[a^2 - \left(V_\infty + \frac{\partial \hat{\varphi}}{\partial x}\right)^2\right]\frac{\partial^2 \hat{\varphi}}{\partial x^2} + \left[a^2 - \left(\frac{\partial \hat{\varphi}}{\partial y}\right)^2\right]\frac{\partial^2 \hat{\varphi}}{\partial y^2} - 2\left(V_\infty + \frac{\partial \varphi}{\partial x}\right)\left(\frac{\partial \hat{\varphi}}{\partial y}\right)\frac{\partial^2 \hat{\varphi}}{\partial x \partial y} = 0$$

(perturbation velocity potential equation)

Note the similarity with the (normal) velocity potential equation! This new equation, in terms of $\hat{\varphi}$, can be used to find a linear approximation to the velocity potential equation, ~~allowing for~~ which can be solved analytically.

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The definition from $\hat{\varphi}$ forms the start of the derivation. In terms of u and v, it can be written as

$$\left[a^2 - (V_\infty + \hat{u})^2\right] \frac{\partial \hat{u}}{\partial x} + (a^2 - \hat{v}^2) \frac{\partial \hat{v}}{\partial y} - 2(V_\infty + \hat{u})\hat{v} \frac{\partial \hat{u}}{\partial y} = 0$$

From the energy equation:

$$\frac{a_\infty^2}{\gamma - 1} + \frac{V_\infty^2}{2} = \frac{a^2}{\gamma - 1} + \frac{(V_\infty + \hat{u})^2 + v^2}{2}$$

$$\Rightarrow (1 - M_\infty^2)\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} = M_\infty^2 \left[(\gamma + 1)\frac{\hat{u}}{V_\infty} + \frac{\gamma+1}{2}\frac{\hat{u}^2}{V_\infty^2} + \frac{\gamma-1}{2}\frac{\hat{v}^2}{V_\infty^2}\right]\frac{\partial \hat{u}}{\partial x}$$
$$+ M_\infty^2 \left[(\gamma - 1)\frac{\hat{u}}{V_\infty} + \frac{\gamma+1}{2}\frac{\hat{v}^2}{V_\infty^2} + \frac{\gamma-1}{2}\frac{\hat{u}^2}{V_\infty^2}\right]\frac{\partial \hat{v}}{\partial y}$$
$$+ M_\infty^2 \left[\frac{\hat{v}}{V_\infty}\left(1 + \frac{\hat{u}}{V_\infty}\right)\left(\frac{\partial \hat{u}}{\partial y} + \frac{\partial \hat{v}}{\partial x}\right)\right]$$

Up to here, the expressions are exact, but also non-linear. Assuming small perturbations ($\frac{\hat{u}}{V_\infty}, \frac{\hat{v}}{V_\infty} \ll 1$ and $\frac{\hat{u}^2}{V_\infty^2}, \frac{\hat{v}^2}{V_\infty^2} \lll 1$), which is equivalent to assuming a slender body at small angle of attack, changes this and allows for simplifying to

$$(1 - M_\infty^2)\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} = 0$$
$$(1 - M_\infty^2)\frac{\partial^2 \hat{\varphi}}{\partial x^2} + \frac{\partial^2 \hat{\varphi}}{\partial y^2} = 0$$

Keep the small perturbations assumption in mind when using this equation. Furthermore, it is only applicable for sub- and supersonic Mach numbers.

When trying to obtain the pressure distribution, use is made of the pressure coefficient:

$$C_p \equiv \frac{p - p_\infty}{q_\infty}$$

$$q_\infty = \frac{1}{2}\rho_\infty V_\infty^2 = \frac{1}{2}\frac{\gamma p_\infty}{\gamma p_\infty}\rho_\infty V_\infty^2 = \frac{\gamma}{2}p_\infty\left(\frac{\rho_\infty}{\gamma p_\infty}\right)V_\infty^2$$
$$= \frac{\gamma}{2}p_\infty a_\infty^2 V_\infty^2$$
$$= \frac{\gamma}{2}p_\infty M_\infty^2$$

$$C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)$$

This equation too has to be linearized, for which we first find a $\frac{p}{p_\infty}$.

$$T + \frac{V^2}{2c_p} = T_\infty + \frac{V_\infty^2}{2c_p} \quad \text{(adiabatic flow, calorically perfect gas)}$$
$$c_p = \frac{\gamma R}{\gamma - 1}$$

$$\Rightarrow T - T_\infty = \frac{V_\infty^2 - V^2}{2\gamma R/(\gamma - 1)}$$
$$\frac{T}{T_\infty} - 1 = \frac{\gamma - 1}{2}\frac{V_\infty^2 - V^2}{2\gamma R T_\infty} = \frac{\gamma - 1}{2}\frac{V_\infty^2 - V^2}{a_\infty^2}$$
$$\frac{T}{T_\infty} = 1 - \frac{\gamma - 1}{2a_\infty^2}\left(2\hat{u}V_\infty + \hat{u}^2 + \hat{v}^2\right)$$
$$\frac{p}{p_\infty} = \left[1 - \frac{\gamma - 1}{2a_\infty^2}\left(2\hat{u}V_\infty + \hat{u}^2 + \hat{v}^2\right)\right]^{\gamma/\gamma - 1} \quad \text{(isentropic flow)}$$
$$= \left[1 - \frac{\gamma - 1}{2}M_\infty^2\left(\frac{2\hat{u}}{V_\infty} + \frac{\hat{u}^2 + \hat{v}^2}{V_\infty^2}\right)\right]^{\gamma/\gamma - 1}$$

Approximating / linearizing this results in

$$\frac{p}{p_\infty} = 1 - \frac{\gamma}{2}M_\infty^2\left(\frac{2\hat{u}}{V_\infty} + \frac{\hat{u}^2 + \hat{v}^2}{V_\infty}\right) + \dots \quad \text{(binomial expansion)}$$

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Substituting into the expression for C_p yields

$$C_p = \frac{2}{\gamma M_\infty^2}\left[1 - \frac{\gamma}{2} M_\infty^2 \left(\frac{2\hat{u}}{V_\infty} + \frac{\hat{u}^2 + \hat{v}^2}{V_\infty^2} + \dots - 1\right)\right]$$

$$= -\frac{2\hat{u}}{V_\infty} - \frac{\hat{u}^2 + \hat{v}^2}{V_\infty^2}$$

$\hookrightarrow \approx 0$

The assumption made in this last step limits the use of this equation to small perturbations (slender body and small α). From the concept of perturbation velocity, an expression for the flow tangency condition can be set up:

$$\tan(\theta) = \frac{v}{u} = \frac{\hat{v}}{\hat{u} + V_\infty} \quad \text{(angle between surface and freestream)}$$

$$= \frac{\hat{v}}{V_\infty} \quad \text{(small perturbations)}$$

$$\frac{\partial\hat{\varphi}}{\partial y} = V_\infty \tan(\theta)$$

p.723-726 Based on the linearized perturbation velocity potential equation is the Prandtl-Glauert compressibility correction, which allows for correcting low-speed incompressible airfoil data to be ~~applic~~ valid in the compressible domain. Assume an airfoil ~~of~~ which shape is given by $y = f(x)$, is thin and is under a small angle of attack. (Hence, assume the conditions used to derive the ~~perturb~~ linearized perturbation velocity potential equation hold.) We define

$$\beta^2 \equiv 1 - M_\infty^2$$

such that the linearized perturbation velocity potential equation (LPVPE) can be written as

$$\beta^2 \frac{\partial^2\hat{\varphi}}{\partial x^2} + \frac{\partial^2\hat{\varphi}}{\partial y^2} = 0$$

Transforming to a new space where $\xi = x$ and $\eta = \beta y$, a new velocity potential can be defined, such that

$$\bar{\varphi}(\xi, \eta) = \beta\,\hat{\varphi}(x, y)$$

Rewriting the LPVPE yields

$$\frac{\partial\hat{\varphi}}{\partial x} = \frac{\partial\hat{\varphi}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial\hat{\varphi}}{\partial \eta}\frac{\partial \eta}{\partial x} \quad \text{(from transformation definition)}$$

$$\frac{\partial\hat{\varphi}}{\partial y} = \frac{\partial\hat{\varphi}}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial\hat{\varphi}}{\partial \eta}\frac{\partial \eta}{\partial y} = \beta\frac{\partial\hat{\varphi}}{\partial \eta}$$

or, in terms of $\bar{\varphi}$,

$$\frac{\partial\hat{\varphi}}{\partial x} = \frac{1}{\beta}\frac{\partial\bar{\varphi}}{\partial \xi}$$

$$\frac{\partial\hat{\varphi}}{\partial y} = \frac{\partial\bar{\varphi}}{\partial \eta}$$

This can be differentiated and substituted into the LPVPE to yield

Laplace's Equation (the governing equation for incompressible flow)

$$\left.\begin{aligned}\frac{\partial^2 \hat{\varphi}}{\partial x^2} &= \frac{1}{\beta}\frac{\partial^2 \bar{\varphi}}{\partial \xi^2}\\ \frac{\partial^2 \hat{\varphi}}{\partial y^2} &= \beta\frac{\partial^2 \bar{\varphi}}{\partial \eta^2}\end{aligned}\right\}$$

$$\Rightarrow \beta^2 \frac{1}{\beta}\frac{\partial^2 \bar{\varphi}}{\partial \xi^2} + \beta\frac{\partial^2 \bar{\varphi}}{\partial \eta^2} = 0$$

We have now related compressible flow in (x,y)-space to incompressible flow in (ξ,η)-space. It can be shown that the airfoil shape is equal for these two flow domains.

The linearized pressure coefficient can also be transformed, resulting in the main result of this derivation:

$$C_p = \frac{-2\hat{u}}{V_\infty} = -\frac{2}{V_\infty}\frac{\partial \hat{\varphi}}{\partial x} = -\frac{2}{V_\infty}\frac{1}{\beta}\frac{\partial \bar{\varphi}}{\partial x} = -\frac{2}{V_\infty}\frac{1}{\beta}\frac{d\bar{\varphi}}{\partial \xi} = \frac{1}{\beta}\left(-\frac{2\bar{u}}{V_\infty}\right)$$

Comparing with the incompressible linear pressure coefficient allows for rewriting:

$$C_p = C_{p,0}\cdot\frac{1}{\beta} = \frac{C_{p,0}}{\sqrt{1-M_\infty^2}} \quad \text{(Prandtl-Glauert rule)}$$

($C_{p,0}$ → incompressible pressure coefficient)

From this, lift and moment coefficients readily follow:

$$C_l = \frac{C_{l,0}}{\sqrt{1-M_\infty^2}}$$

$$C_m = \frac{C_{m,0}}{\sqrt{1-M_\infty^2}}$$

p.727. Since Prandtl and Glauert derived their rule, others have found improved compressibility corrections.

$$C_p = C_{p,0} \Big/ \left[\sqrt{1-M_\infty^2} + \left(M_\infty^2/(1+\sqrt{1-M_\infty^2})\right)C_{p,0}/2\right] \quad \text{(Karman-Tsien rule)}$$

$$= C_{p,0} \Big/ \left[\beta + \left(M_\infty^2/(1+\beta)\right)\cdot C_{p,0}/2\right]$$

$$C_p = C_{p,0} \Big/ \left[\sqrt{1-M_\infty^2} + \left(M_\infty^2\left(1+\frac{\gamma-1}{2}M_\infty^2\right)\Big/2\sqrt{1-M_\infty^2}\right)C_{p,0}\right] \quad \text{(Laitone's rule)}$$

$$= C_{p,0} \Big/ \left[\beta + \left(M_\infty^2\left(1+\frac{\gamma-1}{2}M_\infty^2\right)\Big/2\beta\right)C_{p,0}\right]$$

~~p.728~~ p.729-732 It is well-known that, while the freestream is still subsonic, bubbles of local sonic or supersonic flow can appear. The freestream value for which sonic flow first appears is the critical Mach number, denoted M_{cr}. It can be approximated as follows

$$\left.\begin{aligned}\frac{p_A}{p_\infty} &= \frac{p_A/p_0}{p_\infty/p_0} = \left(\frac{1+(\gamma-1)/2\cdot M_\infty^2}{1+(\gamma-1)/2\cdot M_A^2}\right)^{\gamma/\gamma-1} \quad \text{(isentropic flow)}\\ C_{p,A} &= \frac{2}{\gamma M_\infty^2}\left(\frac{p_A}{p_\infty}-1\right)\end{aligned}\right\}$$

$$\Rightarrow C_{p,A} = \frac{2}{\gamma M_\infty^2}\left[\left(\frac{1+(\gamma-1)/2\cdot M_\infty^2}{1+(\gamma-1)/2\cdot M_A^2}\right)^{\gamma/\gamma-1} - 1\right]$$

When $M_A = 1$ (sonic pressure somewhere along the airfoil), $C_p = C_{p,cr}$

$$C_{p,cr} = \frac{2}{\gamma M_\infty^2}\left[\left(\frac{1+(\gamma-1)/2\cdot M_\infty^2}{1+(\gamma-1)/2}\right)^{\gamma/\gamma-1} - 1\right]$$

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In this last equation, based on $M_A = 1$, $M_\infty = M_{cr}$, the critical Mach number. This equation, and one of the compressibility collections provide a means of estimating M_{cr} for a given airfoil:

1. Find the (incompressible) $C_{p,0}$ at the minimum pressure point (where M will be largest) on the airfoil.
2. Plot C_p (compressible) versus M_∞.
3. Plot $C_{p,cr}$ versus ~~M_∞~~ M_{cr}.
4. The intersection of these two curves will give the critical Mach number.

p.738-739 Associated with supersonic flow is a massive drag increase, which first occurs when the freestream Mach number is slightly above the critical Mach number. This particular M_∞ is called the drag-divergence Mach number, M_{DD}. When M_∞ increases even further, beyond $M_\infty = 1$, the drag decreases again.

p.743 Based on the Prandtl-Glauert compressibility correction is a formula that allows for estimating the lift slope for a (swept) wing in compressible flow, based on the incompressible airfoil (2D!) lift slope a_0:

$$a_{comp} = \frac{a_0 \cos\Lambda}{\sqrt{1 - M_\infty^2 \cos^2\Lambda + [(a_0 \cos\Lambda)/(\pi AR)]^2} + (a_0 \cos\Lambda)/(\pi AR)}$$

p.745-749 Previously, we coped with the supersonic drag increase by using thin airfoils and ~~sw~~ swept wings. Although these tactics are still used, new techniques have emerged. The "area rule" is considered first, and states that the area distribution (cross-sectional area versus distance along the axis of the airplane) should be smooth. Practically, this results in a fuselage cross-section that decreases at the location of the wing.

Supercritical airfoils also help by delaying drag. Rather than trying to increase M_{cr}, these airfoils strive to increase the distance between M_{cr} and M_{DD} (see: page 738, fig. 11.11, distance between c and e). These airfoils have a relatively flat upper surface, which

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ultimately leads to a weaker shock wave, resulting in less drag. The bottom is ~~flat~~ cusped, to compensate for the loss in lift by the ~~fo~~ forward section (flat top leads to negative camber).

p. 750-751 Although useful, the concepts derived and explained so far are useful, they are restricted to thin airfoils at small angles of attack, in an inviscid and irrotational flow with $M_\infty < 0{,}7$. CFD is the way to find flow properties in other regimes.

Aerodynamicists started with solving the nonlinear small-perturbation potential velocity equation for transonic flow

$$(1-M_\infty^2)\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=M_\infty^2\left[(\gamma+1)\frac{\partial\varphi}{\partial x}\frac{1}{V_\infty}\right]\frac{\partial^2\varphi}{\partial x^2}$$

Next, the full potential equation was solved, but the assumption of inviscid flow was still intact. Later again, the Euler equations (full continuity, momentum and energy equations) were solved. Shock waves were now modeled accurately, but viscous flow was not considered: predicting drag was hard or even impossible. Solving the Navier-Stokes equations did allow for that, which is the current state of art. Turbulence, however, remains the Achilles heel.

p.756-758 According to some, Blended Wing Bodies (BWBs) are the future of air transport. They ~~[illegible]~~ make use of a couple of design features that make this concept more efficient than conventional jetliners.

1. Closer approximation of the elliptical lift distribution: the center body airfoil (with larger chord) generates less lift than the outer section, as to keep ~~an~~ a smooth spanwise lift distribution.
2. Supercritical airfoils, to delay M_{∞} and the corresponding increase in drag.
3. The BWB is area-ruled

For two-dimensional, irrotational, isentropic, steady flow of a compressible fluid, the exact velocity potential equation is

$$\left[1-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2}+\left[1-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2}-\frac{2}{a^2}\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial y}\right)\frac{\partial^2\phi}{\partial x\,\partial y}=0 \qquad \textbf{[11.12]}$$

where

$$a^2=a_0^2-\frac{\gamma-1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right] \qquad \textbf{[11.13]}$$

This equation is exact, but it is nonlinear and hence difficult to solve. At present, no general analytical solution to this equation exists.

For the case of small perturbations (slender bodies at low angles of attack), the exact velocity potential equation can be approximated by

$$(1-M_\infty^2)\frac{\partial^2\hat{\phi}}{\partial x^2}+\frac{\partial^2\hat{\phi}}{\partial y^2}=0 \qquad \textbf{[11.18]}$$

This equation is approximate, but linear, and hence more readily solved. This equation holds for subsonic ($0 \le M_\infty \le 0.8$) and supersonic ($1.2 \le M_\infty \le 5$) flows; it dos not hold for transonic ($0.8 \le M_\infty \le 1.2$) or hypersonic ($M_\infty > 5$) flows.

The Prandtl-Glauert rule is a compressibility correction that allows the modification of existing incompressible flow data to take into account compressibility effects:

$$C_p=\frac{C_{p,0}}{\sqrt{1-M_\infty^2}} \qquad \textbf{[11.51]}$$

Also,

$$c_l=\frac{c_{l,0}}{\sqrt{1-M_\infty^2}} \qquad \textbf{[11.52]}$$

and

$$c_m=\frac{c_{m,0}}{\sqrt{1-M_\infty^2}} \qquad \textbf{[11.53]}$$

The critical Mach number is that freestream Mach number at which sonic flow is first obtained at some point on the surface of a body. For thin airfoils, the critical Mach number can be estimated as shown in Figure 11.6.

The drag-divergence Mach number is that freestream Mach number at which a large rise in the drag coefficient occurs, as shown in Figure 11.11.

The area rule for transonic flow states that the cross-sectional area distribution of an airplane, including fuselage, wing, and tail, should have a smooth distribution along the axis of the airplane.

Supercritical airfoils are specially designed profiles to increase the drag-divergence Mach number.

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p.780. Whereas the linearized perturbation velocity potential equation was derived for subsonic and supersonic flow, only the former case has been considered. These results are not applicable to supersonic flow, as the equation dramatically changes:

$$(1-M_\infty^2)\frac{\partial^2\hat{\varphi}}{\partial x^2}+\frac{\partial^2\hat{\varphi}}{\partial y^2}=0$$

$(1-M_\infty^2)$ → >0 for subsonic flow ⇒ elliptic PDE
→ <0 for supersonic flow ⇒ hyperbolic PDE

p.780-783 For supersonic flow, the LPVPE is rewritten as

$$\lambda^2\frac{\partial^2\hat{\varphi}}{\partial x^2}-\frac{\partial^2\hat{\varphi}}{\partial y^2}=0$$

with $\lambda=\sqrt{M_\infty^2-1}$. It is solved by

$$\hat{\varphi}=f(x-\lambda y)$$

$$\Rightarrow \frac{\partial\hat{\varphi}}{\partial x}=f'(x-\lambda y)\frac{\partial(x-\lambda y)}{\partial x} \quad \frac{\partial\hat{\varphi}}{\partial x}=f' \quad \Rightarrow \frac{\partial^2\hat{\varphi}}{\partial x^2}=f''$$

$$\frac{\partial\hat{\varphi}}{\partial y}=f'(x-\lambda y)\frac{\partial(x-\lambda y)}{\partial y} \quad \frac{\partial\hat{\varphi}}{\partial y}=f'(-\lambda) \quad \Rightarrow \frac{\partial^2\hat{\varphi}}{\partial y^2}=\lambda^2 f''$$

Substitution: $\lambda^2 f''-\lambda^2 f''=0$ (q.e.d.)

From this result, we conclude that $\hat{\varphi}$ is constant along lines of constant $x-\lambda y$. Hence

$$\frac{dy}{dx}=\frac{1}{\lambda}=\frac{1}{\sqrt{M_\infty^2-1}}=\tan(\mu)$$

So, $\hat{\varphi}$ is constant along a Mach line. This leads to the realization that (for supersonic flow), any disturbance at the wall cannot propagate upstream. This is in contrast to the situation for subsonic flow, where disturbances propagate everywhere. However, keep in mind that this linearized theorems only hold for small perturbations.

From the above, an expression for the supersonic pressure coefficient can be obtained:

$$\left.\begin{aligned}\hat{u}&=\frac{\partial\hat{\varphi}}{\partial x}=f'\\ \hat{v}&=\frac{\partial\hat{\varphi}}{\partial y}=-\lambda f'\end{aligned}\right\}\ \hat{u}=-\frac{\hat{v}}{\lambda}$$

$$\hat{v}=\frac{\partial\hat{\varphi}}{\partial y}=V_\infty\tan\theta=V_\infty\theta \quad \Big\}\ \hat{u}=-\frac{V_\infty\theta}{\lambda}$$

$$C_p=-\frac{2\hat{u}}{V_\infty}=\frac{2\theta}{\lambda}=\frac{2\theta}{\sqrt{M_\infty^2-1}}$$

C_p → linearized pressure coefficient

This states that "C_p is directly proportional to the local surface inclination with respect to the freestream".

p.704-706 When a surface is inclined into the freestream direction, linearized theory predicts a positive C_p. From this C_p, other useful aerodynamic quantities can be obtained: C_n and C_a, and in turn C_d and C_l.

$$C_{p,l} = \frac{2\alpha}{\sqrt{M_\infty^2 - 1}} \quad \text{(flat plate under } \alpha)$$

$$C_{p,u} = -\frac{2\alpha}{\sqrt{M_\infty^2 - 1}}$$

$$\Rightarrow C_n = \frac{1}{c}\int_0^c (C_{p,l} - C_{p,u})\, dx = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}$$

$$C_a = \frac{1}{c}\int_{LE}^{TE} (C_{p,l} - C_{p,u})\, dy = 0$$

↳ flat plate has zero thickness, dy = 0.

$$\Rightarrow C_l = C_n - C_a\alpha \quad \text{(small } \alpha \text{ approximation)}$$

$$C_d = C_n\alpha + C_a$$

$$\Rightarrow C_l = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}$$

$$C_d = \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}} \quad \text{(wave drag coefficient)}$$

For thin airfoils, the equation for lift coefficient remains valid. To compute wave drag, the previous result has to be changed a little:

$$C_d = \frac{4}{\sqrt{M_\infty^2 - 1}}\left(\alpha^2 + g_c^2 + g_t^2\right)$$

↳ depend on camber and thickness

In linearized supersonic flow, information is propagated along Mach lines where the Mach angle $\mu = \sin^{-1}(1/M_\infty)$. Since these Mach lines are all based on M_∞, they are straight, parallel lines which propagate away from and downstream of a body. For this reason, disturbances cannot propagate upstream in a steady supersonic flow.

The pressure coefficient, based on linearized theory, on a surface inclined at a small angle θ to the freestream is

$$C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}} \quad \textbf{[12.15]}$$

If the surface is inclined into the freestream, C_p is positive; if the surface is inclined away from the freestream, C_p is negative.

Based on linearized supersonic theory, the lift and wave-drag coefficients for a flat plate at an angle of attack are

$$c_l = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}} \quad \textbf{[12.23]}$$

and

$$c_d = \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}} \quad \textbf{[12.24]}$$

Equation (12.23) also holds for a thin airfoil of arbitrary shape. However, for such an airfoil, the wave-drag coefficient depends on both the shape of the mean camber line and the airfoil thickness.