Two-Dimensional Airfoils

1 Definitions

There are various ways to describe an airfoil. The NACA-terminology is a well-known standard, which defines the following airfoil properties. The **mean camber line** is the line formed by the points halfway between the upper and lower surfaces of the airfoil. The most forward and rearward points of the airfoil are the **leading edge** and the **trailing edge**, respectively. The straight line connecting the leading and trailing edges is the **chord line**.

The length of the chord line is defined as the **chord** c. The maximum distance between the chord line and the camber line is called the **camber**. If the camber is 0, then the airfoil is called **symmetric**. And finally, the **thickness** is the distance between the upper and lower surfaces of the airfoil.

In this chapter we will be looking at 2-dimensional airfoils. We're interested in finding c_l, the **lift coefficient per unit length**. At low **angles of attack** α, the value of c_l varies linearly with α. The **lift slope** a_0 is the ratio of them, so $a_0 = \frac{dc_l}{d\alpha}$.

If α gets too high, this relation doesn't hold, since **stall** will occur. The maximum value of c_l is denoted by $c_{l,max}$. This value determines the minimum velocity of an aircraft. The value of α when $c_l = 0$ is called the **zero-lift angle of attack** and is denoted by $\alpha_{L=0}$.

2 Vortex sheets

In the last chapter we treated the source panel method. We put a lot of sources on a sheet. We can also put a lot of vortices on a curve s. Let's define $\gamma = \gamma(s)$ as the strength of the vortex sheet per unit length along s. The velocity potential at some point P can then be determined, using

$$d\phi = -\frac{\gamma\, ds}{2\pi}\theta \qquad \Rightarrow \qquad \phi = -\frac{1}{2\pi}\int_a^b \theta\gamma\, ds. \tag{2.1}$$

Here θ is the angle between point P and the point on the vortex sheet we're at that moment looking at. Also a and b are the begin and the end of the vortex sheet.

The circulation of the vortex sheet can be determined to be

$$\Gamma = \int_a^b \gamma\, ds. \tag{2.2}$$

If the circulation is known, the resulting lift can be calculated using the Kutta-Joukowski theorem

$$L' = \rho_\infty V_\infty \Gamma. \tag{2.3}$$

3 Kutta condition

We can put a vortex sheet on the camber line of an airfoil. We can then use boundary conditions and numerical computation to find the vortex strength γ at every point. But it turns out that there are multiple solutions. To get one solution, we can use the **Kutta condition**, which states that the flows leaves the trailing edge smoothly.

What can we derive from this? For now, let's call φ the angle of the trailing edge. Also let's call V_1 the velocity on top of the airfoil at the trailing edge and V_2 the velocity at the bottom of the airfoil at the same point. If φ is finite, then it can be shown that $V_1 = V_2 = 0$. However, if $\varphi \to 0$ (the trailing edge is cusped), then only $V_1 = V_2$. Nevertheless, we can derive the same rule from both situations. Namely, that the vortex strength at the trailing edge is

$$\gamma(TE) = 0. \tag{3.1}$$

4 Thin airfoil theory

Suppose we want to calculate the flow over a very thin airfoil by using a vortex sheet in a free stream flow. We can put vortices on the camber. But the camber line doesn't differ much from the chord line, so to keep things simple we place vortices on the chord line.

Since the airfoil is thin, it is by itself a streamline of the flow. So the velocity perpendicular to the camber line is 0. Let's define $z(x)$ to be the distance between the mean camber line and the chord line, where x is the distance from the leading edge. The velocity perpendicular to the camber line, caused by the free stream flow, at position x, can be shown to be

$$V_{\infty,n} = V_\infty \left(\alpha - \frac{dz}{dx}\right), \tag{4.1}$$

where α is in radians. The velocity perpendicular to the mean camber line, due to the vortices, is approximately equal to the velocity perpendicular to the chord. It can be shown that this velocity component on a small part $d\varepsilon$, with distance x from the airfoil leading edge, is

$$dw = -\frac{\gamma(\varepsilon)\, d\varepsilon}{2\pi(x-\varepsilon)}. \tag{4.2}$$

Integrating along the chord gives the total velocity perpendicular to the chord at position x due to the vortex sheet, being

$$w(x) = -\frac{1}{2\pi}\int_0^c \frac{\gamma(\varepsilon)\, d\varepsilon}{x-\varepsilon}. \tag{4.3}$$

We have already derived that the velocity perpendicular to the airfoil is zero. So $V_{\infty,n} + w = 0$, which results in

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\varepsilon)\, d\varepsilon}{x-\varepsilon} = V_\infty \left(\alpha - \frac{dz}{dx}\right). \tag{4.4}$$

This is the **fundamental equation of thin airfoil theory**.

5 Vortex distributions of symmetric airfoils

If we have a symmetric airfoil, then there is no camber, so $dz/dx = 0$ everywhere on the airfoil. This simplifies equation 4.4 and we might actually try to solve it now. If we make the change of variable $\varepsilon = \frac{1}{2}c(1-\cos\theta)$ and also define $x = \frac{1}{2}c(1-\cos\theta_0)$, we get

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\, d\theta}{\cos\theta - \cos\theta_0} = V_\infty \alpha. \tag{5.1}$$

This is a complicated integral, but it can be solved. The solution will be

$$\gamma(\theta) = 2\alpha V_\infty \frac{1+\cos\theta}{\sin\theta}. \tag{5.2}$$

We might want to take a closer look on the change of variable we have made. How can we visualize this change of variable? Imagine the airfoil being the diameter of a circle. Now imagine we are moving over the top half of the circle, from the leading edge to the trailing edge. The angle θ we make with respect to the center of the airfoil corresponds to the point on the airfoil directly below it, as is shown in figure 1.

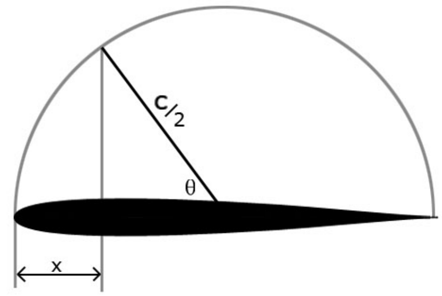

Figure 1: Clarification of the change of variable.

6 Lift coefficients of symmetric airfoils

In the last paragraph, we found the vortex strength of a thin symmetric airfoil. Using the vortex strength, we can find the circulation, which will turn out to be

$$\Gamma = \pi \alpha c V_\infty. \tag{6.1}$$

Using the Kutta-Joukowski theorem, we can calculate the lift per unit span on the airfoil, which is

$$L' = \rho_\infty V_\infty \Gamma = \pi \alpha c \rho_\infty V_\infty^2. \tag{6.2}$$

The lift coefficient now is

$$c_l = \frac{L'}{\frac{1}{2}\rho_\infty V_\infty^2 c} = \frac{\pi \alpha c \rho_\infty V_\infty^2}{\frac{1}{2}\rho_\infty V_\infty^2 c} = 2\pi\alpha. \tag{6.3}$$

So we know have the important conclusion that for thin symmetric airfoils, the lift slope is $a_0 = 2\pi$.

7 Moment coefficients of symmetric airfoils

We can use this theory as well to calculate the moment per unit span exerted on the airfoil around, for example, the leading edge. Let's call M' the moment per unit span around the leading edge. Moment is force times distance, so $dM' = -\varepsilon dL'$. The minus sign is there due to sign convention. We know that the lift per unit span is $L' = \rho_\infty V_\infty \Gamma$, so we find that $dL' = \rho_\infty V_\infty d\Gamma$. We also know that $d\Gamma = \gamma(\varepsilon) d\varepsilon$. Combining this all gives

$$M'_{LE} = -\int_0^c \varepsilon \, dL' = -\rho_\infty V_\infty \int_0^c \varepsilon \gamma(\varepsilon) \, d\varepsilon. \tag{7.1}$$

Using the familiar change of variable and integrating gives

$$M'_{LE} = -q_\infty \left(\frac{c}{2}\right)^2 2\pi\alpha = -c_l q_\infty \left(\frac{c}{2}\right)^2. \tag{7.2}$$

The **moment coefficient about the leading edge** now is

$$c_{m,le} = \frac{M'_{LE}}{q_\infty c^2} = -\frac{c_l q_\infty \left(\frac{c}{2}\right)^2}{q_\infty c^2} = -\frac{1}{4} c_l. \tag{7.3}$$

The **quarter-chord point** is the point at distance $\frac{1}{4}c$ from the leading edge. Taking sum of the moments about the quarter-chord point gives the **moment coefficient about the quarter-chord point**

$$c_{m,c/4} = c_{m,le} + \frac{1}{4} c_l = 0. \tag{7.4}$$

The **center of pressure** is the point around which there is no moment. So the center of pressure is equal to the quarter-chord position. The **aerodynamic center** is the point around which the moment coefficient is independent of α. Since $c_{m,c/4} = 0$ for every α, the quarter point position is also the aerodynamic center. So the center of pressure and the aerodynamic center are both located at the quarter-chord point.

8 Vortex distributions of cambered airfoils

For cambered airfoils, it is a lot more difficult to solve equation 4.4, since $\frac{dz}{dx} \neq 0$. Mathematicians have found the solution to be

$$\gamma(\theta) = 2V_\infty \left(A_0 \frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta \right). \tag{8.1}$$

We will not show the derivation, since that will be too complicated. You will just have to accept the equations.

The values A_n depend on $\frac{dz}{dx}$ and A_0 depends on both $\frac{dz}{dx}$ and α. In fact, using even more complicated mathematics, it can be shown that

$$A_0 = \alpha - \frac{1}{\pi}\int_0^\pi \frac{dz}{dx} d\theta_0, \qquad A_n = \frac{2}{\pi}\int_0^\pi \frac{dz}{dx}\cos n\theta_0 \, d\theta_0. \tag{8.2}$$

Note that $\frac{dz}{dx}$ is the derivative of $z(x)$, taken at point x. So the value of $\frac{dz}{dx}$ depends on x. And x also depends on θ_0, since $x = \frac{1}{2}c(1-\cos\theta)$.

9 Lift coefficients of cambered airfoils

Let's take a loot at the lift coefficient of the airfoil. The circulation can be found using

$$\Gamma = cV_\infty \left(\pi A_0 + \frac{\pi}{2} A_1 \right). \tag{9.1}$$

The lift per unit span now is

$$L' = \rho_\infty V_\infty \Gamma = \rho_\infty V_\infty^2 c\pi \left(A_0 + \frac{1}{2} A_1 \right). \tag{9.2}$$

The lift coefficient can be shown to be

$$c_l = \frac{L'}{\frac{1}{2}\rho_\infty V_\infty^2 c} = \pi(2A_0 + A_1) = 2\pi \left(\alpha + \frac{1}{\pi}\int_0^\pi \frac{dz}{dx}(\cos\theta_0 - 1) d\theta_0 \right). \tag{9.3}$$

We now see that the lift slope is once more $a_0 = \frac{dc_l}{d\alpha} = 2\pi$. So camber does not change the lift slope. However, it does change the zero-lift angle of attack, which will be

$$\alpha_{L=0} = 2\pi\alpha - c_l = -\frac{1}{\pi}\int_0^\pi \frac{dz}{dx}(\cos\theta_0 - 1) d\theta_0. \tag{9.4}$$

10 Moment coefficients of cambered airfoils

Just like we did for symmetric airfoils, we can calculate the moment coefficient. The result will be

$$c_{m,le} = -\frac{\pi}{2}\left(A_0 + A_1 - \frac{A_2}{2} \right) = -\frac{c_l}{4} + \frac{\pi}{4}(A_2 - A_1). \tag{10.1}$$

We can once more derive the moment coefficient with respect to the quarter-chord point. It will not be 0 this time, but

$$c_{m,c/4} = \frac{\pi}{4}(A_2 - A_1). \tag{10.2}$$

The value of $c_{m,c/4}$ is independent of α, so the quarter-chord point is the aerodynamic center. However, the moment coefficient is not zero, so this point is not the center of pressure. The position of the center of pressure can be calculated to be

$$x_{cp} = -\frac{M'_{LE}}{L'} = -\frac{c_{m,le}c}{c_l} = \frac{c}{4}\left(1 + \frac{\pi}{c_l}(A_1 - A_2)\right) \tag{10.3}$$

11 Designing a camber line

We used the camber line (described by $\frac{dz}{dx}$) to find the coefficients $A_0, A_1, \ldots$. We can also use the coefficients to find the camber line. We then have several boundary conditions. Of course $z(0) = 0$ and $z(c) = 0$.

First we need to think of suitable coefficients for our design. What these coefficients will be depends on what properties we want to give our airfoil. For example, if we want to have $c_{m,c/4} = 0$, then we should take $A_1 = A_2$. If we have determined our coefficients, we can find our camber line by using

$$\frac{dz}{dx} = \alpha - A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta_0. \tag{11.1}$$

12 Design lift coefficient

Thin airfoils do have a disadvantage. For most angles of attack, the airflow separates at the leading edge (and reattaches afterward for low velocities). This reduces lift. For one angle of attack, the flow smoothly attaches to the leading edge. This is the so-called **ideal** or **optimal angle of attack** α_{opt}.

Theoretical calculations can show that this only occurs if the vortex at the leading edge is zero, so $\gamma_{LE} = 0$. Combining this fact with equation 8.1 gives $A_0 = 0$. Inserting this in equation 8.2 results in

$$\alpha_{opt} = \frac{1}{\pi}\int_0^{\pi} \frac{dz}{dx}\theta_0. \tag{12.1}$$

The lift coefficient at the optimal angle of attack is called the **design lift coefficient**. Thanks to equation 9.3, we can calculate it, using

$$(c_l)_{design} = \pi A_1 = 2\int_0^{\pi} \frac{dz}{dx}\cos\theta_0\, d\theta_0. \tag{12.2}$$