Three-Dimensional Wings

1 Induced drag

So far we have looked at two-dimensional (infinite) wings. Now let's look at three-dimensional wings. Lift is created by a high pressure on the bottom of the wing and a low pressure on top of the wing. At the wing edges, air tries to go from the bottom to the top of the wing. This causes **vortices**.

These vortices cause a small velocity component in the downward direction at the wing, called **downwash**. So the airfoil "sees" a different flow direction than the free stream flow. Even though α is the **geometric angle of attack** (with respect to the free stream flow), the **effective angle of attack** α_{eff}, which actually contributes to the lift, is different. This is such that

$$\alpha_{eff} = \alpha - \alpha_i, \tag{1.1}$$

where α_i is the change of the direction of the air flow close to the airfoil. α_i is called the **induced angle of attack**.

But the decrease in lift is only small. The real disadvantage is that the lift factor is tilted backward by an angle α_i. So part of the "lift" is pointing in the direction of the free stream flow, so it is actually drag. This drag is called **induced drag**.

2 Coefficients, lift and drag

In the last chapter we have dealt with the lift coefficient per unit span c_l. Now we will deal with the actual lift coefficient C_L of the entire wing. Identically, the lift per unit span L' becomes the total lift L. Also c_d becomes C_D and D' becomes D. The same goes for moment coefficients.

In real life, the drag consists of three parts. There is **skin friction drag** D_f, **pressure drag** D_p and induced drag D_i. The first two are caused by viscous effect, and together form the **profile drag**. If $C_{D,p}$ is the **profile drag coefficient**, then

$$C_{D,p} = \frac{D_f + D_p}{q_\infty S}. \tag{2.1}$$

The **induced drag coefficient** $C_{D,i}$ is

$$C_{D,i} = \frac{D_i}{q_\infty S}. \tag{2.2}$$

Together they form the total drag coefficient, being

$$C_D = \frac{D_f + D_p + D_i}{q_\infty S} = C_{D,p} + C_{D,i}. \tag{2.3}$$

3 Vortex filaments

In the last chapter, we considered 2-dimensional vortices. We can put a lot of them in a three-dimensional curve, being a so-called **vortex filament**. A vortex filament has a strength Γ. If we now look at any part of the curve $\mathbf{dl}$, then the velocity $\mathbf{dV}$ at some point P, caused by this part, is

$$\mathbf{dV} = \frac{\Gamma}{4\pi}\frac{\mathbf{dl} \times \mathbf{r}}{|\mathbf{r}|^3}, \tag{3.1}$$

where $\mathbf{r}$ is the vector from the part $\mathbf{dl}$ to point P. This important relation is called the **Biot-Savart law**.

There are a few important rules concerning vortex filaments. These are called **Helmholtz's vortex theorems**.

- The strength of a vortex filament is constant along its entire length.
- A vortex filament can not end. It is either a closed curve or it is infinitely long.

4 Horseshoe vortices

We can model three-dimensional wings using vortex filaments. Let's take a wing with wing span b, and put it in a coordinate system such that the tips are positioned at $y = -\frac{b}{2}$ and $y = \frac{b}{2}$. Now we can let a vortex filament run from one tip to the other. But a vortex filament may not end. So from the tip, we let the filaments (both ends) run to infinity in the direction of the free stream flow (which is defined as the positive x-direction). The part of the vortex filament on the wing is called the **bound vortex**. The two infinite parts are the **trailing vortices**. The entire vortex filament is called a **horseshoe vortex**, since it has the shape of a horseshoe (except for the fact that horses don't have infinite feet).

Using the horseshoe vortex, we can already, more or less, model the wing. But if we go to the wing tips, the induced velocity will go to infinity, which isn't what happens in real life. So we need to change our model. Instead of having one horseshoe vortex, running between $-\frac{b}{2}$ and $\frac{b}{2}$, we put infinitely many, running between $-y$ and y, where $0 \leq y \leq \frac{b}{2}$. We now have a vortex distribution $\Gamma(y)$ along the wing and a vortex sheet with strength $d\Gamma(y)$ behind the wing.

5 Induced angle of attack

Now look at a point on the wing with y-coordinate y_0. The velocity induced by the semi-infinite trailing vortex at position y can be found using the Biot-Savart law. The result will be

$$dw = -\frac{\left(\frac{d\Gamma}{dy}\right) dy}{4\pi(y_0 - y)}. \tag{5.1}$$

So if we want to find the entire induced velocity at point y_0, we need to integrate along the entire wing, giving

$$w(y_0) = -\frac{1}{4\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\left(\frac{d\Gamma}{dy}\right) dy}{(y_0 - y)}. \tag{5.2}$$

Using the induced velocity, we can find the induced angle of attack to be

$$\alpha_i(y_0) = \tan^{-1}\left(\frac{-w(y_0)}{V_\infty}\right) \approx -\frac{w(y_0)}{V_\infty} = \frac{1}{4\pi V_\infty}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\left(\frac{d\Gamma}{dy}\right) dy}{(y_0 - y)}. \tag{5.3}$$

Note that w is defined positive upward, but the induced angle of attack was defined to be positive directed downward. Therefore a minus sign is present. It is also assumed that w is small with respect to V_∞, so the small angle approximation can be used.

6 Finding the vortex distribution

Now let's derive some more expressions. From the previous chapter, we know that the lift coefficient per unit span at the point y_0 is

$$c_l = a_0(\alpha_{eff}(y_0) - \alpha_{L=0}), \tag{6.1}$$

where $a_0 = 2\pi$ for thin wings. But the lift coefficient can also be found using

$$L' = \frac{1}{2}\rho_\infty V_\infty^2 c(y_0) c_l = \rho_\infty V_\infty \Gamma(y_0) \qquad \Rightarrow \qquad c_l = \frac{2\Gamma(y_0)}{V_\infty c(y_0)}. \tag{6.2}$$

Combining these equations and solving for α_{eff} gives

$$\alpha_{eff} = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0}. \tag{6.3}$$

If we put everything together, the angle of attack can be calculated. The result is

$$\alpha = \alpha_{eff} + \alpha_i = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0} + \frac{1}{4\pi V_\infty} \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\left(\frac{d\Gamma}{dy}\right) dy}{(y_0 - y)}. \tag{6.4}$$

This equation is the **fundamental equation of Prandtl's lifting-line theory**. For a wing with a given design, all values are known except Γ. So this is in fact a differential equation with which Γ can be found.

If Γ is found, we can find the lift distribution using the Kutta-Joukowski theorem ($L'(y) = \rho_\infty V_\infty \Gamma$). Also the induced drag distribution can be found by using

$$D_i'(y) = L' \sin \alpha_i \approx L' \alpha_i = \rho_\infty V_\infty \Gamma \alpha_i. \tag{6.5}$$

From the lift and drag distribution, the total lift and drag can be found, by integrating over the wing (from $-\frac{b}{2}$ to $\frac{b}{2}$).

7 Elliptical lift distribution

Suppose we have a wing with a circulation distribution given by

$$\Gamma(y) = \Gamma_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2}, \tag{7.1}$$

where Γ_0 is (per definition) the circulation at $y = 0$. It can now be shown that the induced velocity and induced angle of attack are

$$w = -\frac{\Gamma_0}{2b} \qquad \Rightarrow \qquad \alpha_i = -\frac{w}{V_\infty} = \frac{\Gamma_0}{2bV_\infty}. \tag{7.2}$$

Now let's define the **aspect ratio** as

$$A = \frac{b^2}{S}. \tag{7.3}$$

If we first express Γ as a function of C_L, fill it in in equation 7.2 and use the definition of the aspect ratio, then we can derive that

$$\alpha_i = \frac{C_L}{\pi A} \qquad \Rightarrow \qquad C_{D,i} = \frac{C_L^2}{\pi A}. \tag{7.4}$$

So the induced drag only depends on the lift coefficient and the aspect ratio. Long slender wings thus give low induced drag.

8 General lift distribution

Let's suppose we don't know Γ. If we make the change-of-variable $y = -\frac{b}{2}\cos\theta$, we can use a lot of complicated mathematics to transform equation 6.4 to

$$\alpha(\theta) = \frac{2b}{\pi c(\theta)} \sum_{1}^{N} A_n \sin n\theta + \alpha_{L=0}(\theta) + \sum_{1}^{N} n A_n \frac{\sin n\theta}{\sin\theta}. \tag{8.1}$$

In this equation, the coefficients $A_1, \ldots, A_n$ are the unknown coefficients that need to be determined. If N is higher (so if there are more coefficients), the result will be more precise. Do not mix up the coefficients A_i and the aspect ratio A.

To find $A_1, \ldots, A_N$, you have to apply the equation at N points on the wing. Then you have N equations and N unknowns, which can be solved. You can take any N points on the wing, except for the tips, since $\Gamma = 0$ at those positions.

We can also derive the lift coefficient to be

$$C_L = A_1 \pi A. \tag{8.2}$$

If we work things out a lot more, we get an expression for the drag coefficient, which appears very familiar. The result is

$$C_{D,i} = \frac{C_L^2}{\pi A e}. \tag{8.3}$$

The number e is called **oswald's factor** and is defined as

$$e = A_1^2 \left(\sum_{1}^{N} n A_n^2 \right)^{-1} = \frac{A_1^2}{A_1^2 + 2A_2^2 + \ldots + nA_n^2}. \tag{8.4}$$

It is clear that $e \leq 1$ (with $e = 1$ only if $A_i = 0$ for $i \geq 2$). For an elliptical lift distribution $e = 1$, so this lift distribution is the distribution with the lowest induced drag. However, to minimize induced drag, it is often more important to worry about the aspect ratio, then about the lift distribution.