Supersonic Flow over Airfoils

In the previous chapter we treated subsonic flow over airfoils. In this final chapter we will take a look at supersonic flow. How do airfoils behave at $M > 1$?

1 The Linearized Supersonic Pressure Coefficient Equation

In the previous chapter, we derived the linear perturbation velocity potential equation. If we define $\lambda = \sqrt{M_\infty^2 - 1}$, we can rewrite it to

$$\lambda^2 \frac{\partial^2 \hat{\phi}}{\partial x^2} - \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0. \tag{1.1}$$

Any function $\hat{\phi} = f(x - \lambda y)$ satisfies this equation. So it initially may not seem helpful. However, we do know that if $x - \lambda y = \text{constant}$, also $\hat{\phi}$ stays constant. Also, $x - \lambda y$ is constant, if

$$\frac{dy}{dx} = \frac{1}{\lambda} = \frac{1}{\sqrt{M_\infty^2 - 1}} = \tan \mu, \tag{1.2}$$

where μ is the **Mach angle**, which was introduced in the chapter on oblique shock waves. So we find that $\hat{\phi}$ is constant along a **Mach line**.

From the fact that $\hat{\phi} = f(x - \lambda y)$, we can also derive another important relation. From this follows that, for a certain position on the wing with angle θ, we have

$$\hat{u} = -\frac{V_\infty \theta}{\lambda}. \tag{1.3}$$

The pressure coefficient can now be found using

$$C_p = -\frac{2\hat{u}}{V_\infty} = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}. \tag{1.4}$$

This important equation is called the **linearized supersonic pressure coefficient equation**. It is a rather simple way to find C_p. The sign of θ, and thus also of C_p can, however, be rather tricky. Luckily you only have to remember one important thing. If the surface of the airfoil is inclined into the free stream, there is a relatively high pressure, and C_p is thus positive. On the other hand, if the surface is inclined away from the free stream, the pressure is relatively low, and C_p is thus negative.

2 Lift and Drag Coefficients of a Flat Plate

Let's give an example of how to use the relation that was just derived. Let's calculate the lift and drag coefficient of a flat plate at an angle of attack α in a supersonic flow. The pressure coefficients at the lower and upper side of the plate, $C_{p,l}$ and $C_{p,u}$, respectively, are given by

$$C_{p,l} = \frac{2\alpha}{\sqrt{M_\infty^2 - 1}} \qquad \text{and} \qquad C_{p,u} = -\frac{2\alpha}{\sqrt{M_\infty^2 - 1}}. \tag{2.1}$$

The component of the force acting normal to the plate c_n can now be found using

$$c_n = \frac{1}{c} \int_0^c \left(C_{p,l} - C_{p,u}\right) dx = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}. \tag{2.2}$$

Since the plate has no thickness, there is no component of the force acting parallel to the plate. So we have

$$c_l = c_n \cos\alpha \qquad \text{and} \qquad c_d = c_n \sin\alpha. \tag{2.3}$$

Using $\cos\alpha \approx 1$ and $\sin\alpha \approx \alpha$ we eventually get

$$c_l = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}} \qquad \text{and} \qquad c_n = \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}}. \tag{2.4}$$

These equations are only valid for flat plates at small angles of attack. Supersonic airplanes, however, usually have relatively flat wings, and also fly at low angles of attack. So the above equations can often also be applied for the wings of supersonic aircrafts. Isn't it surprising that such simple equations can say so much about such complicated aircrafts?