Subsonic Compressible Flow over Airfoils

It is time to turn theory into practice. What can we say about flow over airfoils? In this chapter we consider compressible subsonic flow over airfoils. The next chapter focusses on supersonic flow.

1 The Velocity Potential Equation

In a previous aerodynamics course we have seen the velocity potential ϕ. It was defined such that

$$\mathbf{V} = \nabla\phi. \tag{1.1}$$

From the velocity potential we can find the velocity components

$$u = \frac{\partial\phi}{\partial x} \qquad \text{and} \qquad v = \frac{\partial\phi}{\partial y}. \tag{1.2}$$

For incompressible flows (ρ is constant) we would get **Laplace's equation**

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{1.3}$$

This equation is a linear differential equation. There exist solutions for it. If ρ is not constant, things are a lot more difficult. Using (among others) the continuity equation and Euler's equation, we can eventually derive that

$$\left(1 - \frac{1}{a^2}\left(\frac{\partial\phi}{\partial x}\right)^2\right)\frac{\partial^2\phi}{\partial x^2} + \left(1 - \frac{1}{a^2}\left(\frac{\partial\phi}{\partial y}\right)^2\right)\frac{\partial^2\phi}{\partial y^2} - \frac{2}{a^2}\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\,\partial y} = 0. \tag{1.4}$$

This important equation is called the **velocity potential equation**. Note that for incompressible flows we would have ρ constant and thus $a = \infty$. The above equation then reduces back to Laplace's equation. However, a is not infinite. It also depends on the velocity potential. This is according to

$$a^2 = a_0^2 - \frac{\gamma - 1}{2}\left(\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right), \tag{1.5}$$

where a_0 is constant for the entire flow.

2 The Linearized Velocity Potential Equation

The velocity potential is a nonlinear equation. It is therefore very hard to solve. To solve it, we have to use assumptions, through which we can turn the above equation into a linear equation.

But before we do that, we have to introduce the **perturbation velocities** $\hat{u}$ and $\hat{v}$. Let's suppose we are flying with a **free stream velocity** V_∞ in x-direction. The velocity perturbations are now defined as the change in velocity, with respect to the free stream velocity. So

$$\hat{u} = u - V_\infty \qquad \text{and} \qquad \hat{v} = v. \tag{2.1}$$

Identically, we can define the **perturbation velocity potential** $\hat{\phi}$ such that

$$\hat{u} = \frac{\partial\hat{\phi}}{\partial x} \qquad \text{and} \qquad \hat{v} = \frac{\partial\hat{\phi}}{\partial y}. \tag{2.2}$$

Using this perturbation velocity, we can derive a very complicated equation, similar to the velocity potential equation. However, for certain **free stream Mach numbers** M_∞, this equation can be simplified. If $0 \leq M_\infty \leq 0.8$ or $M_\infty \geq 1.2$, certain parts can be neglected. If also $M_\infty < 5$ even more parts can be neglected. We also have to make the assumption that the velocity perturbations $\hat{u}$ and $\hat{v}$ are small. This is usually the case for thin bodies at small angles of attack. Based on all these assumptions, the complicated equation reduces to

$$\left(1 - M_\infty^2\right) \frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0. \tag{2.3}$$

This is the **linearized perturbation velocity potential equation**. It is a linear partial differential equation. With it, the perturbation velocity potential can be found. However, when doing that, we also need boundary conditions. There are two boundary conditions that can be used. First of all, at $x = \infty$, we have $\hat{u} = \hat{v} = 0$ and thus $\hat{\phi} = $ constant. Second, we know that if the airfoil is at an angle θ with respect to the free stream flow, then also

$$\frac{\partial \hat{\phi}}{\partial y} = \hat{v} = (V_\infty + \hat{u}) \tan\theta \approx V_\infty \tan\theta. \tag{2.4}$$

So now we know how to find $\hat{\phi}$. What can we do with it? Well, with it we can find the pressure coefficient. The **pressure coefficient** could normally be found using

$$C_p = \frac{p - p_\infty}{q_\infty} = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right). \tag{2.5}$$

For small velocity perturbations the ratio p/p_∞ can be approximated by

$$\frac{p}{p_\infty} = 1 - \frac{\gamma}{2} M_\infty^2 \left(\frac{2\hat{u}}{V_\infty} + \frac{\hat{u}^2 + \hat{v}^2}{V_\infty^2}\right). \tag{2.6}$$

Using this, the relation for the pressure coefficient can be simplified to

$$C_p = -\frac{2\hat{u}}{V_\infty}. \tag{2.7}$$

3 Compressibility Corrections

Instead of deriving entirely new equations for compressible flows, we can also slightly change existing equations for incompressible flows, such that they approximate compressible flows. Such adjustments are called **compressibility corrections**.

The first compressibility correction is the **Prandtl-Glauert correction**. It stated that the pressure coefficient C_p in a compressible flow can be derived from the pressure coefficient $C_{p,0}$ in an incompressible flow, according to

$$C_p = \frac{C_{p,0}}{\sqrt{1 - M_\infty^2}}. \tag{3.1}$$

The lift coefficient and moment coefficient for compressible flow can be derived similarly, using

$$c_l = \frac{c_{l,0}}{\sqrt{1 - M_\infty^2}} \qquad \text{and} \qquad c_m = \frac{c_{m,0}}{\sqrt{1 - M_\infty^2}}. \tag{3.2}$$

The Prandtl-Glauert rule is based on the linearized velocity potential equation. Other compressibility corrections do take the nonlinear terms into account. Examples are the **Karman-Tsien** rule, which states that

$$C_p = \frac{C_{p,0}}{\sqrt{1 - M_\infty^2} + \frac{M_\infty^2}{1 + \sqrt{1 - M_\infty^2}} \frac{C_{p,0}}{2}}, \tag{3.3}$$

and Laitone's rule, stating that

$$C_p = \frac{C_{p,0}}{\sqrt{1-M_\infty^2} + M_\infty^2\left(1+\frac{\gamma-1}{2}M_\infty^2\right)\sqrt{1-M_\infty^2}\,\frac{C_{p,0}}{2}}. \tag{3.4}$$

4 The Critical Mach Number

The flow velocity is different on different positions on the wing. Let's say we know the Mach number M_A of the flow over our wing at a given point A. The corresponding pressure coefficient can then be found using

$$C_{p,A} = \frac{2}{\gamma M_\infty^2}\left(\left(\frac{1+\frac{\gamma-1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}M_A^2}\right)^{\frac{\gamma}{\gamma-1}} - 1\right). \tag{4.1}$$

The velocity of the flow on top of our wing is generally bigger than the free stream velocity V_∞. So we may have sonic flow ($M = 1$) over our wing, while we are still flying at $M_\infty < 1$. The **critical Mach number** M_{cr} is defined as the free stream Mach number M_∞ at which sonic flow ($M = 1$) is first achieved on the airfoil surface. It is a very important value. To find it, we use the **critical pressure coefficient** $C_{p,cr}$. The relation between M_{cr} and $C_{p,cr}$ can be found from the above equation. This relation is

$$C_{p,cr} = \frac{2}{\gamma M_{cr}^2}\left(\left(\frac{1+\frac{\gamma-1}{2}M_{cr}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma}{\gamma-1}} - 1\right). \tag{4.2}$$

However, the above equation has two unknowns. So we need an additional equation. We can use any of the compressibility corrections for that. But, to do that, we first need to know $C_{p,0}$. This can be found using low-speed wind tunnel tests. Just measure the minimal pressure coefficient over the entire wing. This is the position of minimum pressure and thus maximum velocity. So once $C_{p,0}$ is known, we have two equations with two unknowns. It can be solved.

5 The Increase in Drag

You may wonder, why is the critical Mach number so important? We can see that if we plot the drag coefficient c_d with respect to the free stream Mach number M_∞. Initially c_d has the constant value of $c_{d,0}$. However, when M_∞ gets bigger than M_{cr}, shock waves will appear. This causes additional drag. So the critical Mach number relates to the velocity at which the drag increases.

At a certain free stream Mach number the drag coefficient suddenly starts to increase enormously. The Mach number at which this occurs (which is often slightly bigger than M_{cr}) is called the **drag-divergence Mach number**. However, once we have passed $M_\infty = 1$, the drag coefficient c_d decreases. We have then passed the so-called **sound barrier**.

Normally, the drag coefficient can get as big as ten times $c_{d,0}$. There are, however, ways to prevent this. One way is the so-called **area rule**. It seems that sudden changes in the cross-sectional area of a wing cause a high drag coefficient at the sonic region. So, the cross-sectional area of an airplane should be as constant as possible. At the positions of the wings, the cross-section of the aircraft usually increases. To prevent this, the cross-section of the fuselage at those points should decrease. This can reduce the drag coefficient at $M = 1$ by an entire factor 2.

Another way of reducing the drag around $M = 1$ is by using **supercritical airfoils**. The idea behind this is not to increase the critical Mach number M_{cr}. Instead, it is to increase the drag-divergence Mach number. This is done by making the top of the airfoil as flat as possible. By doing this, airplanes can fly at higher velocities, without experiencing the massive increase in drag just yet.