CHAPTER 3

3.1 Consider steady, inviscid flow.

x-momentum: $$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} = - \frac{\partial p}{\partial x} \quad (1)$$

y-momentum: $$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \rho w \frac{\partial v}{\partial z} = - \frac{\partial p}{\partial y} \quad (2)$$

z-momentum: $$\rho u \frac{\partial w}{\partial x} + \rho v \frac{\partial w}{\partial y} + \rho w \frac{\partial w}{\partial z} = - \frac{\partial p}{\partial z} \quad (3)$$

Multiply (1), (2), and (3) by dx, dy, and dz respectively:

$$u \frac{\partial u}{\partial x} dx + v \frac{\partial u}{\partial y} dx + w \frac{\partial u}{\partial z} dx = - \frac{1}{\rho} \frac{\partial p}{\partial x} dx \quad (4)$$

$$u \frac{\partial v}{\partial x} dy + v \frac{\partial v}{\partial y} dy + w \frac{\partial v}{\partial z} dy = - \frac{1}{\rho} \frac{\partial p}{\partial y} dy \quad (5)$$

$$u \frac{\partial w}{\partial x} dz + v \frac{\partial w}{\partial y} dz + w \frac{\partial w}{\partial z} dz = - \frac{1}{\rho} \frac{\partial p}{\partial y} dz \quad (6)$$

Add (4) + (5) + (6):

$$u\left(\frac{\partial u}{\partial x} dx + \frac{\partial v}{\partial x} dy + \frac{\partial w}{\partial x} dz\right) + v \left(\frac{\partial u}{\partial y} dx + \frac{\partial v}{\partial y} dy + \frac{\partial w}{\partial y} dz\right)$$

$$+ w \left(\frac{\partial u}{\partial z} dx + \frac{\partial v}{\partial z} dy + \frac{\partial w}{\partial z} dz\right) = - \frac{1}{\rho} \left(\frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz\right) \quad (7)$$

For irrotational flow (see Eq. (2.119)): $\nabla \times V = 0$

Hence:

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}; \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}; \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \quad (8)$$

Subt. Eqs. (8) into (7):

$$u\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\right) + v\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz\right)$$

$$+ w\left(\frac{\partial w}{\partial z}dx + \frac{\partial w}{\partial z}dy + \frac{\partial w}{\partial z}dz\right) = -\frac{1}{\rho}\left(\frac{\partial p}{\partial z}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right)$$

$$u\,du + v\,dv + w\,dw = -\frac{1}{\rho}\,dp$$

$$\frac{1}{2}d\,(u^2 + v^2 + w^2) = \frac{1}{2}d\,(V^2) = VdV = -\frac{1}{\rho}\,dp$$

$dp = -\rho V\,dV$ which integrates to

$$\boxed{p + \frac{1}{2}\rho V^2 = \text{const.}}$$

for incompressible flow.

3.2 $$V_1 = \sqrt{\frac{(2)(p_1 - p_2)}{\rho\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]}}$$

$p_1 = 2116$ lb/ft^2; $p_2 = 2100$ lb/ft^2, $A_2/A_1 = 0.8$

$$V_1 = \sqrt{\frac{2(2116 - 2100)}{(0.002377)\left[\left(\frac{1}{0.8}\right)^2 - 1\right]}} = \boxed{154.7 \text{ ft/sec}}$$

3.3 $$p_1 - p_2 = \tfrac{1}{2}\rho V_1^2\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right] = \tfrac{1}{2}(1.23)(90)^2\,[(1/0.85)^2 - 1] = \boxed{1913 \text{ N/m}^2}$$

3.4 $$V_2 = \sqrt{\frac{2\ w\ \Delta h}{\rho\left[1 - (A_2/A_1)^2\right]}}$$

$$w = \rho_m g = (1.36 \times 10^4)\left(9.8 \frac{m}{sec^2}\right) = 1.33 \times 10^5 \text{ N/m}^2$$

$$\Delta h = 10 \text{ cm} = 0.1\text{m};\ \rho = 1.23 \text{ kg/m}^3,\ \frac{A_2}{A_1} = \frac{1}{12}$$

$$V_2 = \sqrt{\frac{2(1.33 \times 10^5)(0.1)}{(1.23)\left[1 - \left(\frac{1}{12}\right)^2\right]}} = \boxed{147 \text{ m/sec}}$$

3.5 $p_1 - p_2 = w\,\Delta h = (1.33 \times 10^5)(0.1) = 1.33 \times 10^4 \text{ N/m}^2$

$$p_2 = p_1 - 1.33 \times 10^4 = 1.01 \times 10^5 - 1.33 \times 10^4 = 8.77 \times 10^4 \text{ N/m}^2$$

$$p_o = p_2 + \frac{1}{2}\rho V_2^2 = 8.77 \times 10^4 + \frac{1}{2}(1.23)(147)^2 = \boxed{1.01 \times 10^5 \text{ N/m}^2}$$

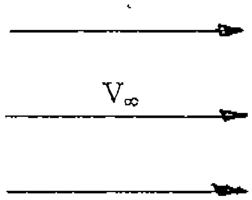
Note: It makes sense that the total pressure in the test section would equal one atmosphere, because the flow in the tunnel is drawn directly from the open ambient surroundings, and for an inviscid flow, we have no losses between the inlet and the test section.

3.6 $p_o = p_\infty + \frac{1}{2}\rho V_\infty^2$

$$V_\infty = \sqrt{\frac{2(p_o - p_\infty)}{\rho}} = \sqrt{\frac{2(1.07 - 1.01) \times 10^5}{1.23}} = \boxed{98.8 \frac{m}{sec}}$$

3.7 $C_p = 1 - \left(\frac{V}{V_\infty}\right)^2 = 1 - \left(\frac{130}{98.8}\right)^2 = \boxed{-0.73}$

3.8



$$\vec{V} = V_\infty \vec{i} \qquad V_\infty = u = \text{constant}$$

$$\nabla \cdot \vec{V} = \overset{0}{\cancel{\frac{\partial u}{\partial x}}} + \overset{0}{\cancel{\frac{\partial v}{\partial y}}} + \overset{0}{\cancel{\frac{\partial w}{\partial z}}} = \boxed{0}$$

It is a physically possible incompressible flow.

$$\nabla \times V = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & \cancelto{0}{v} & \cancelto{0}{w} \end{vmatrix} = \vec{i}\,(0\text{-}0) - \vec{j}\,(0 - \cancelto{0}{\frac{\partial v}{\partial x}}) + \vec{k}\,(0 - \cancelto{0}{\frac{\partial v}{\partial y}})$$

$\boxed{\nabla \times \vec{V} = 0}$ The flow is irrotational.

3.9 For a source flow,

$$\vec{V} = V_r\,\vec{e}_r = \frac{\Lambda}{2\pi r}\,\vec{e}_r$$

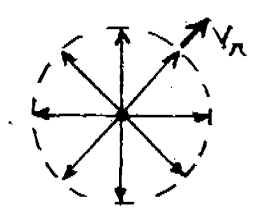
In polar coordinates:

$$\nabla \cdot \vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(r\,V_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}$$

$$\nabla \cdot \vec{V} = \frac{1}{r}\frac{\partial}{\partial r}\,[r\,\frac{\Lambda}{2\pi r}] + \frac{1}{r}\frac{\partial (0)}{\partial \theta}$$

$$\nabla \cdot \vec{V} = \cancelto{0}{\frac{1}{r}\frac{\partial}{\partial r}(\frac{\Lambda}{2\pi})} + 0 = \boxed{0}$$

Hence, the flow is a physical possible incompressible flow, except at the origin where $r = 0$.



What happens at the origin? Visualize a cylinder of radius r wrapped around the line source per unit depth perpendicular to the page. The volume flow across this cylindrical surface is

$$\oiint_S \vec{V} \cdot \vec{dS} \tag{1}$$

Since we are considering a unit depth, then we have the volume flow per unit depth. This is precisely the definition of source strength, Λ. Hence, from (1),

$$\Lambda = \text{constant} = \oiint_S \vec{V} \cdot \vec{dS} \tag{2}$$

From the divergence theorem:

$$\oiint_S \vec{V} \cdot \vec{dS} = \oiiint_V (\nabla \cdot \vec{V})\, dV \tag{3}$$

Combining Eqs. (2) and (3)

$$\oiiint_V (\nabla \cdot \vec{V})\, dV = \Lambda = \text{constant} \tag{4}$$

Shrink the volume to an infinitesimal value, ΔV, around the origin. Eq. (4) becomes

$$(\nabla \cdot \vec{V})\, \Delta V = \Lambda$$

Taking the limit as $\Delta V \to 0$

$$(\nabla \cdot \vec{V}) = \lim \frac{\Lambda}{\Delta V} = \infty.$$

Hence $\nabla \cdot \vec{V} = \infty$ at origin

To show that the flow is irrotational, calculate $\nabla \mathbf{x} \vec{V}$.

$$\nabla \times \vec{V} = \frac{1}{r} \begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & \vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ V_r & rV_\theta & V_z \end{vmatrix} = \frac{1}{r} \begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & \vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \frac{\Lambda}{2\pi r} & 0 & 0 \end{vmatrix}$$

$$\nabla \times \vec{V} = -r\, \vec{e}_\theta \left(\cancelto{0}{\frac{\partial 0}{\partial r}} - \cancelto{0}{\frac{\partial \Lambda / 2\pi r}{\partial z}} \right) + \vec{e}_z \left(\cancelto{0}{\frac{\partial 0}{\partial r}} - \cancelto{0}{\frac{\partial \Lambda / 2\pi r}{\partial \theta}} \right) = \boxed{0}$$

Hence,

$$\boxed{\nabla \times \vec{V} = 0 \text{ everywhere.}}$$

3.10

$$\phi = V_\infty x; \qquad \frac{\partial \phi}{\partial x} = V_\infty \,; \quad \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\frac{\partial \phi}{\partial y} = 0 \;; \quad \frac{\partial^2 \phi}{\partial y^2} = 0$$

Hence, Laplaces equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 + 0 = 0 \text{ is identically satisfied.}$$

Similarly, for $\psi = V\,y$; $\frac{\partial \psi}{\partial x} = 0, \ \frac{\partial^2 \psi}{\partial x^2} = 0$

$$\frac{\partial \psi}{\partial y} = V, \ \frac{\partial^2 \psi}{\partial y^2} = 0$$

Hence, Laplaces equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 + 0 = 0 \text{ is identically satisfied.}$$

3.11 $\phi = \frac{\Lambda}{2\pi} \ell n \; r; \ \frac{\partial \phi}{\partial r} = \frac{\Lambda}{2\pi}\frac{1}{r}, \ \frac{\partial^2 \phi}{\partial r^2} = -\frac{\Lambda}{2\pi}\frac{1}{r^2}$

$$\frac{\partial \phi}{\partial \theta} = 0 \,, \ \frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Hence, Laplace's equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\Lambda}{\partial \pi}\right] + 0 = 0$$

O

is identically satisfied.

$$\psi = \frac{\Lambda}{2} = \theta \ ; \quad \frac{\partial \psi}{\partial r} = 0 \ . \quad \frac{\partial^2 \psi}{\partial r^2} = 0$$

$$\frac{\partial \psi}{\partial \theta} = \frac{\Lambda}{2\pi} \ , \ \frac{\partial^2 \psi}{\partial \phi^2} = 0$$

Hence, Laplaces equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = \frac{1}{r}\frac{\partial}{\partial r}(0) + \frac{1}{r^2}(0) = 0$$

is identically satisfied.

3.12 The stagnation point is a distance $\Lambda/2\pi V_\infty$ upstream of the source. Hence,

$$\frac{\Lambda}{2\pi V_\infty} = 1 \ , \text{ or } \ \Lambda = 2\pi V_\infty$$

The shape of the body is given by

$$\psi = V_\infty r \sin\theta + \frac{\Lambda}{2\pi}\ \theta = \frac{\Lambda}{2}$$

or,

$$r \sin\theta + \frac{\Lambda}{2\pi V_\infty}\theta = \frac{\Lambda}{2V_\infty}$$

or,

$$r \sin\theta + \frac{2\pi V_\infty}{2\pi V}\theta = \frac{2\pi V_\infty}{2V_\infty}$$

or,

$$\boxed{r \sin\theta + \theta = \pi} \qquad \text{Equation of the semi-infinite body.}$$

$$r = \frac{\pi - \theta}{\sin\theta}$$

θ(rad)	r	x = r cosθ	y = r sinθ
π	1	-1	0
3	1.0033	-0.990	0.1416
2.8	1.02	-0.961	0.3416
2.5	1.072	-0.859	0.6416
2.0	1.255	-0.522	1.142
π/2	1.57	0	1.57
1.3	1.91	0.511	1.84
1.0	2.54	1.372	2.14
0.75	3.509	2.57	2.39
0.5	5.51	4.84	2.64

Cartesian Coordinates of Body

To plot the pressure coefficient:

$$V_r = V_\infty \cos\theta + \frac{\Lambda}{2\pi r} = V_\infty \cos\theta + \frac{2\pi V_\infty}{2\pi r} = V_\infty \cos\theta + \frac{V_\infty}{r}$$

$$V_\theta = -V_\infty \sin\theta$$

or,

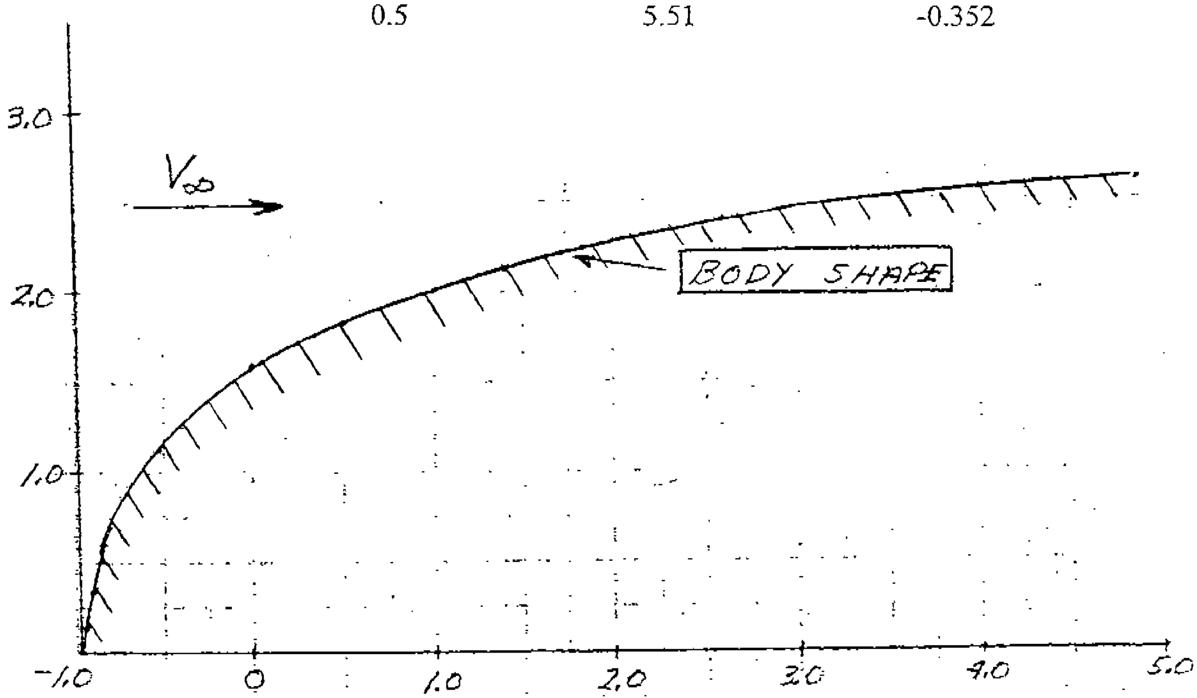
$$\frac{V_r}{V_\infty} = \cos\theta + \frac{1}{r}$$

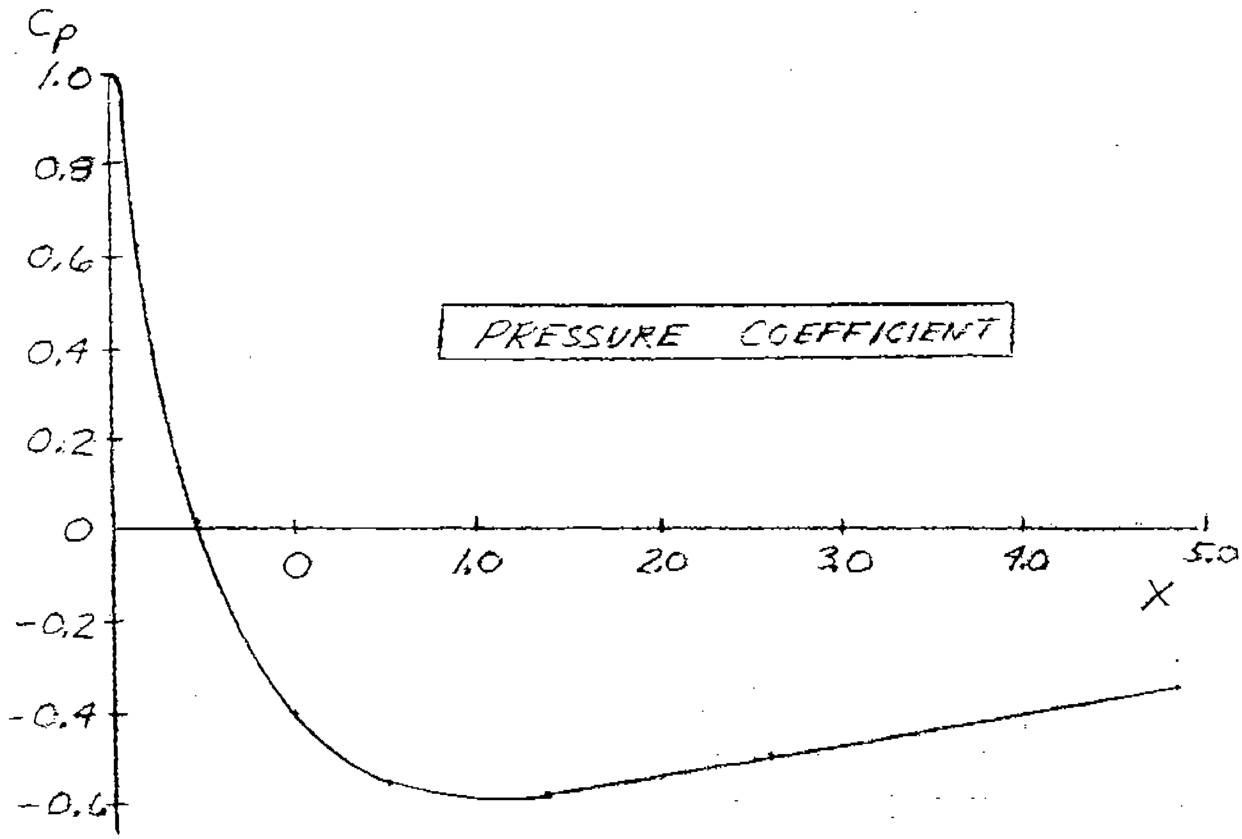
$$\frac{V_\theta}{V_\infty} = -\sin\theta$$

$$\left(\frac{V}{V_\infty}\right)^2 = \left(\frac{V_r}{V_\infty}\right)^2 + \left(\frac{V_\theta}{V_\infty}\right)^2 = \cos\theta + \frac{2}{r}\cos\theta + \frac{1}{r^2} + \sin^2\theta = 1 + \frac{2}{r}\cos\theta + \frac{1}{r^2}$$

$$C_p = 1 - \left(\frac{V}{V_\infty}\right)^2 = -\frac{2}{r}\cos\theta - \frac{1}{r^2}$$

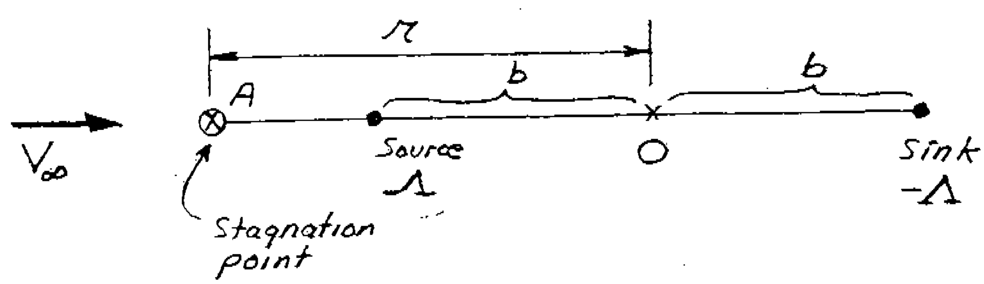
θ(rad)	r	C_p
π	1	1.0
3	1.00	0.98
2.8	1.02	0.886
2.5	1.072	0.624
2.0	1.255	0.0283
π/2	1.57	-0.4057
1.3	1.91	-0.554
1.0	2.54	-0.580
0.75	3.509	-0.4982
0.5	5.51	-0.352





Cp
1.0
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
PRESSURE COEFFICIENT
0
1.0
2.0
3.0
4.0
5.0
X

3.13



r
b
b
A
V∞
Source
Λ
O
Sink
-Λ
Stagnation point

At point A: Velocity due to freestream = V_∞

Velocity due to source = $\frac{-\Lambda}{2\pi(r+b)}$

(note that it is in the negative x-direction)

Velocity due to sink = $\frac{(+\Lambda)}{2\pi(r+b)}$

(Note that it is in the positive x-direction)

Total velocity at Point A:

$$V_A = V_\infty - \frac{\Lambda}{2\pi}\frac{1}{(r-b)} + \frac{\Lambda}{2\pi}\frac{1}{(r+b)}$$

From point A to be a stagnation point, $V_A = 0$.

$$0 = V_\infty + \frac{\Lambda}{2\pi}\left[\frac{1}{(r+b)} + \frac{1}{(r-b)}\right].$$

$$0 = V_\infty + \frac{\Lambda}{2\pi}\left[\frac{r-b-(r+b)}{(r+b)(r-b)}\right] = V_\infty + \frac{\Lambda}{2\pi}\frac{(-2b)}{r^2-b^2}$$

$$V_\infty(r^2-b^2) = \frac{\Lambda}{2\pi}(2b) = \frac{\Lambda b}{\pi}$$

$$r^2 = \frac{\Lambda b}{\pi V_\infty} + b^2$$

$$\boxed{r = \sqrt{\frac{\Lambda b}{\pi V_\infty} + b^2}}$$

3.14 $V_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$ (1)

For a doublet: $\psi = -\frac{k}{2\pi}\frac{\sin\theta}{r}$

$$\frac{\partial \psi}{\partial \theta} = -\frac{\kappa}{2\pi}\frac{\cos\theta}{r} \qquad (2)$$

Substitute (2) into (1)

$$\frac{\partial \phi}{\partial r} = \frac{1}{r}\left(-\frac{\kappa}{2\pi}\frac{\cos\theta}{r}\right) = -\frac{\kappa}{2\pi}\frac{\cos\theta}{r^2}$$

Integrating with respect to r

$$\phi = \left(-\frac{\kappa}{2\pi}\cos\theta\right)\left(-\frac{1}{r}\right)$$

or,

$$\phi = \frac{\kappa}{2\pi}\frac{\cos\theta}{r}$$

3.15 $\psi = (V_\infty \, r \sin\theta)\left(1 - \frac{R^2}{r^2}\right)$

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = (V_\infty \cos\theta)\left(1 - \frac{R^2}{r^2}\right)$$

$$V_\theta = -\frac{\partial \psi}{\partial r} = -\left(1 + \frac{R^2}{r^2}\right) V_\infty \sin\theta$$

$$V^2 = V_r^2 + V_\theta^2 = \left(1 - \frac{R^2}{r^2}\right)^2 V_\infty^2 \cos^2\theta + \left(1 + \frac{R^2}{r^2}\right)^2 V_\infty \sin^2\theta$$

$$C_p = 1 - \frac{V^2}{V_\infty^2} = 1 - \left(1 - \frac{R^2}{r^2}\right)^2 \cos^2\theta - \left(1 - \frac{R^2}{r^2}\right)^2 \sin^2\theta$$

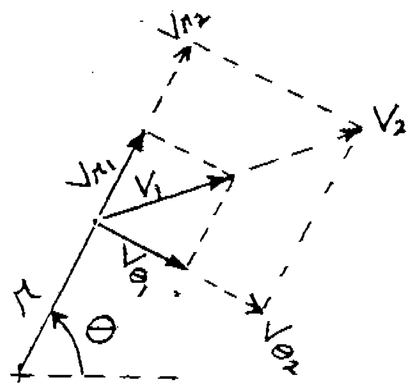
At the surface, r = R

$$C_p = 1 - 4\sin^2\theta$$

3.16 From Eq. (3.93): $\frac{V_r}{V_\infty} = \left(1 - \frac{R^2}{r^2}\right)\cos\theta$

From Eq. (3.94): $$\frac{V_\theta}{V_\infty} = -\left(1+\frac{R^2}{r^2}\right)\sin\theta$$

At any given point (r,θ), V_r and V_θ are both directly proportional to V_∞. Hence, the <u>direction</u> of the resultant, $\vec{V}$, is the same, no matter what the value of V_∞ may be. Thus, the <u>shape of the streamlines remains the same</u>.



3.17 From Eq. (3.119): $$\frac{V_r}{V_\infty} = \left(1-\frac{R^2}{r^2}\right)\cos\theta$$

From Eq. (3.94): $$\frac{V_\theta}{V_\infty} = -\left(1+\frac{R^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r V_\infty}$$

Note that V_θ/V_∞ is itself a function of V_∞ via the second term. Hence, as V_∞ changes, the direction of the resultant velocity at a given point will also change. The shape of the <u>streamlines changes</u> when V_∞ changes.

3.18 $L' = \rho_\infty V_\infty \Gamma$

$$\Gamma = \frac{L'}{\rho_\infty V_\infty} = \frac{6}{(1.23)(30)} = \boxed{0.163\ m^2/sec}$$

3.19 At standard sea level conditions,

$$\rho_\infty = 0.002377 \ \frac{\text{slug}}{\text{ft}^3}$$

$$\mu_\infty = 3.737 \times 10^{-7} \ \frac{\text{slug}}{(\text{ft})(\text{sec})}$$

Also:

$$V = 120 \text{ mph} = 120 \left(\frac{88}{60}\right) \text{ ft/sec} = 176 \ \frac{\text{ft}}{\text{sec}}$$

$$q_\infty = \frac{1}{2} \rho_\infty V_\infty^2 = \frac{1}{2} (0.002377)(176)^2 = 36.8 \text{ lb/ft}^2$$

For the struts: D = 2 in = 0.167 ft.

$$\text{Re} = \frac{\rho V D}{\mu} = \frac{(0.002377)(187.7)(0.167)}{3.737 \times 10^{-7}} = 199{,}382$$

From Fig. 3.39, $C_D = 1$. The total frontal surface area of the struts is (25) (0.167) = 4.175 ft^2. Hence,

Drag due to struts:

$$D_S = q_\infty S C_D = (36.8)(4.175)(1) = 153 \text{ lb}$$

For the bracing wires: $D = \frac{3}{32}$ in = 0.0078 ft

$$\text{Re} = 199382 \left(\frac{0.0078}{0.167}\right) = 9312$$

From Fig. 3.39, $C_D = 1$. The total frontal surface area of the wires is (80) (0.0078) = 0.624 ft^2. Hence,

Drag due to wires:

$$D_w = q_\infty S C_D = (36.8)(0.624)(1) = 23 \text{ lb}$$

Total drag due to struts and wires = $D_S + D_W$ =

$$153 + 23 = \boxed{176}$$

The total zero-lift drag for the airplane is (including struts and wires)

$$C_{D_o} = q_\infty \, S \, C_{D_o} = (36.8)(230)(0.036) = \boxed{304.8}$$

Note that, for this example, the drag due to the struts and wires is $\frac{176}{304.8} = 0.58$ of the total drag – i.e., 58 percent of the total drag. This clearly points out the drag reduction that was achieved in the early 1930's when airplane designers started using internally braced wings with one or more central spars, thus eliminating struts and wires completely.

3.20 The flow over the airfoil in Figure 3.37 can be syntheized by a proper distribution of singularities, i.e., point sources and vortices. The strength of the vortices, added together, gives the total circulation, Γ, around the airfoil. This value of Γ is the same along all closed curves around the airfoil, even if the closed curve is drawn a very large distance away from the airfoil. In this case, the airfoil becomes a speck on the page, and the distributed point vortices appear as one stronger point vortex with strength Γ. This is exactly equivalent to the single point vortex in Figure 3.27 for the circular cylinder, and the lift on the airfoil where the circulation is taken as the total Γ is the same as for a circular cylinder, namely Eq. (3.140),

$$L' = \rho_\infty V_\infty \Gamma$$