Relations between height, pressure, density and temperature

1 Definitions

g = Gravitational acceleration at a certain altitude ($g_0 = 9.81 m/s^2$) (m/s^2)

r = Earth radius ($6378km$) (m)

h_g = Height above the ground (Geometric height) (m)

h_a = Height above the center of the earth ($h_a = h_g + r$) (m)

h = Geopotential altitude (Geopotential height) (m)

p = Pressure ($Pa = N/m^2$)

ρ = Air density (kg/m^3)

$\nu = \frac{1}{\rho}$ = Specific volume (m^3/kg)

T = Temperature (K)

$R = 287.05 J/(kgK)$ = Gas constant

$P_s = 1.01325 \times 10^5 N/m^2$ = Pressure at sea level

$\rho_s = 1.225 kg/m^3$ = Air density at sea level

$T_s = 288.15K$ = Temperature at sea level

$a = \frac{dT}{dh}$ = Temperature gradient ($a = 0.0065K/m$ in the troposphere (lowest part) of the earth atmosphere) (K/m)

2 Relation between geopotential height and geometric height

Newton's gravitational law implicates:

$$g = g_0 \left(\frac{r}{h_a}\right)^2 = g_0 \left(\frac{r}{r + h_g}\right)^2$$

The hydrostatic equation is:

$$dp = -\rho g dh_g$$

However, g is variable here for different heights. Since a variable gravitational acceleration is difficult to work with, the geopotential height h has been introduced such that:

$$dp = -\rho g_0 dh \tag{2.1}$$

So this means that:

$$dh = \frac{g}{g_0} dh_g = \frac{r^2}{(r + h_g)^2} dh_g$$

And integration gives the general relationship between geopotential height and geometric height:

$$h = \frac{r}{r + h_g} h_g \tag{2.2}$$

3 Relations between pressure, density and height

The famous equation of state is:

$$p = \rho R T \tag{3.1}$$

Dividing the hydrostatic equation (2.1) by the equation of state (3.1) gives as results:

$$\frac{dp}{p} = \frac{-\rho g_o dh}{\rho R T} = -\frac{g_o}{RT} dh$$

If we assume an isothermal environment (the temperature stays the same), then integration gives:

$$\int_{p_1}^{p_2} \frac{dp}{p} = -\frac{g_o}{RT} \int_{h_1}^{h} dh$$

Solving this gives the following equation:

$$\frac{p_2}{p_1} = e^{-\left(\frac{g_0}{RT}\right)(h_2 - h_1)} \tag{3.2}$$

And combining this with the equation of state gives the following equation:

$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1} \frac{TR}{TR} = \frac{p_2}{p_1} = e^{-\left(\frac{g_0}{RT}\right)(h_2 - h_1)} \tag{3.3}$$

4 Relations between pressure, density and temperature

We now again divide the hydrostatic equation (2.1) by the equation of state (3.1), but this time we don't assume an isothermal environment, but we substitute $dh = \frac{dT}{a}$ in it, to get:

$$\frac{dp}{p} = \frac{-\rho g_o dh}{\rho R T} = -\frac{g_o}{aR} \frac{dT}{T}$$

Integration gives:

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{-\frac{g_0}{aR}} \tag{4.1}$$

Which is a nice formula. But by using the equation of state, we can also derive the following:

$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1} \frac{T_1 R}{T_2 R} = \frac{p_2}{p_1} \left(\frac{T_2}{T_1}\right)^{-1} = \left(\frac{T_2}{T_1}\right)^{-\frac{g_0}{aR}} \left(\frac{T_2}{T_1}\right)^{-1} = \left(\frac{T_2}{T_1}\right)^{-\left(\frac{g_0}{aR}+1\right)} \tag{4.2}$$

All those relations can be written in a simpler way:

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\frac{g_0}{g_0 + aR}} = \left(\frac{T_2}{T_1}\right)^{-\frac{g_0}{aR}} \tag{4.3}$$

These relations are the standard atmospheric relations in gradient layers.