Normal Shock Waves

Where there are supersonic flows, there are usually also shock waves. A fundamental type of shock wave is the **normal shock wave** – the shock wave normal to the flow direction. We will examine that type of shock wave in this chapter.

1 Basic Relations

Let's consider a rectangular piece of air (the system) around a normal shock wave, as is shown in figure 1. To the left of this shock wave are the initial properties of the flow (denoted by the subscript 1). To the right are the conditions behind the wave.

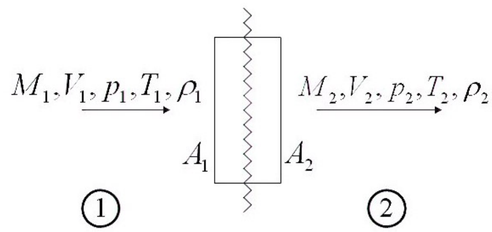

Figure 1: A normal shock wave.

We can already note a few things about the flow. It is a steady flow (the properties stay constant in time). It is also adiabatic, since no heat is added. No viscous effects are present between the system and its boundaries. Finally, there are no body forces.

Now what can we derive? Using the continuity equation, we can find that the mass flow that enters the system on the left is $\rho_1 u_1 A_1$, with u the velocity of the flow in x-direction. The mass flow that leaves the system on the right is $\rho_2 u_2 A_2$. However, since the system is rectangular, we have $A_1 = A_2$. So we find that

$$\rho_1 u_1 = \rho_2 u_2. \tag{1.1}$$

We can also use the momentum equation. The momentum entering the system every second is given by $(\rho_1 u_1 A_1) u_1$. The momentum flow leaving the system is identically $(\rho_2 u_2 A_2) u_2$. The net force acting on the system is given by $p_1 A_1 - p_2 A_2$. Combining everything, we can find that

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2. \tag{1.2}$$

Finally let's look at the energy. The energy entering the system every second is $(\rho_1 u_1 A_1) \left(e_1 + u_1^2/2\right)$. Identically, the energy leaving the system is $(\rho_2 u_2 A_2) \left(e_2 + u_2^2/2\right)$. No heat is added to the system (the flow is adiabatic). There is work done on the system though. The amount of work done every second is $p_1 A_1 u_1 - p_2 A_2 u_2$. Once more, we can combine everything to get

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}. \tag{1.3}$$

This equation states that the total enthalpy is the same on both sides of the shock wave. Since the shock wave was adiabatic, we actually already knew that. So this was no surprising result.

The three equations we have just derived hold for all one-dimensional, steady, adiabatic, inviscid flows. But let's take a closer look at them. Let's suppose that all upstream conditions ρ_1, u_1, p_1, h_1 and T_1 are known. We can't solve for all the downstream conditions just yet. We have only three equations, while we have four unknowns. We need a few more equations. These equations are

$$h = c_p T, \tag{1.4}$$

$$p = \rho R T. \tag{1.5}$$

That wasn't much new, was it? We now have 5 unknowns and 5 equations. So we can solve everything.

2 The Speed of Sound

A special kind of normal shock wave is a sound wave. In fact, it is an infinitesimally weak normal shock wave. Because of this, dissipative phenomena (like viscosity and thermal conduction) can be neglected, making it an isentropic flow.

At what velocity does this shock wave travel? Let's call this velocity the **speed of sound** a. Note that $a = u_1$. Because the shock wave is very weak, we can also state that $p_2 = p_1 + dp$, $\rho_2 = \rho_1 + d\rho$ and $a_2 = a_1 + da$. If we combine these facts with the three equations we derived in the previous paragraph, we eventually find that

$$a^2 = \frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s. \tag{2.1}$$

The last part in the above equation is to indicate that the changes in p and ρ occur isentropically. For isentropic processes we have

$$p = c\rho^\gamma \qquad \Rightarrow \qquad \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p}{\rho}. \tag{2.2}$$

This results in

$$a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma R T}, \tag{2.3}$$

where we used the equation of state in the last part. So apparently, for a given medium, the speed of sound only depends on the temperature.

Do you still remember the compressibility we introduced in the previous chapter? From the equation $d\rho = \rho \tau dp$, we can also derive that

$$a = \sqrt{\frac{1}{\rho \tau_s}}. \tag{2.4}$$

Note that we have used the isentropic compressibility because the process is isentropic. So we see that the lower the compressibility of a substance, the faster sound travels in it.

3 The Mach Number

The Mach number M is defined as

$$M = \frac{u}{a}. \tag{3.1}$$

A lot of properties can be derived from the Mach number. Let's recall the total temperature T_0. This can be found using

$$c_p T_0 = c_p T + \frac{u^2}{2}. \tag{3.2}$$

From this we can derive that

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2. \tag{3.3}$$

Using the isentropic flow relations, we can also find that

$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}, \tag{3.4}$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma-1}}. \tag{3.5}$$

From equation (3.2) we can also derive that

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{a_0^2}{\gamma - 1} = \text{constant}. \tag{3.6}$$

4 Sonic Conditions

When you slow an airflow down adiabatically to $u = 0$ (and thus $M = 0$) you find the total temperature T_t, total pressure p_t, total density ρ_t, and so on. Similarly, we can change the velocity of a flow adiabatically such that $M = 1$. The corresponding **temperature at sonic conditions** is denoted by T^*. The **characteristic speed of sound** a^* can now be found using $a^* = \sqrt{\gamma R T^*}$. However, we can also determine that

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{a^{*2}}{\gamma - 1} + \frac{a^{*2}}{2} = \frac{\gamma + 1}{2(\gamma - 1)} a^{*2} = \text{constant}. \tag{4.1}$$

Just like we can examine the speed of sound at sonic conditions, we can also look at the temperature T^*, pressure p^* and density ρ^* at such conditions. By inserting $M = 1$ in equations (3.3) to (3.5) we find that

$$\frac{T_0}{T^*} = \frac{\gamma + 1}{2}, \qquad \frac{p_0}{p^*} = \left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma}{\gamma-1}} \qquad \text{and} \qquad \frac{\rho_0}{\rho^*} = \left(\frac{\gamma + 1}{2}\right)^{\frac{1}{\gamma-1}}. \tag{4.2}$$

Finally we can define the **characteristic Mach number** M^* as

$$M^* = \frac{u}{a^*}. \tag{4.3}$$

We can find that M and M^* are related, according to

$$M^2 = \frac{2M^{*2}}{(\gamma + 1) - (\gamma - 1) M^{*2}} \qquad \Leftrightarrow \qquad M^{*2} = \frac{(\gamma + 1) M^2}{2 + (\gamma - 1) M^2}. \tag{4.4}$$

The parameters M and M^* are quite similar. If one is bigger than 1, so is the other, and vice verse.

5 Normal Shock Wave Relations

There are several other relations that hold for normal shock waves. We will discuss some of them. We start with the **Prandtl relation**, stating that

$$a^{*2} = u_1 u_2 \qquad \Leftrightarrow \qquad 1 = M_1^* M_2^*. \tag{5.1}$$

From this follows that

$$M_2^2 = \frac{2 + (\gamma - 1) M_1^2}{2\gamma M_1^2 - (\gamma - 1)}. \tag{5.2}$$

This is an important relation. If $M_1 > 1$ we have $M_2 < 1$. If $M_1 = 1$, then also $M_2 = 1$. (If this is the case we are dealing with an infinitely weak shock wave, called a **Mach wave**.) However, if $M_1 < 1$ it would seem that $M_2 > 1$. But this seems rather odd. Suddenly a subsonic flow becomes supersonic! A more detailed look would show that in this case also the entropy s would decrease. But the second law of thermodynamics states that the entropy can only increase. What can we conclude from this? It means that in subsonic flows no shock waves can appear. Shock waves are thus only present in supersonic flows.

Now we know how to find M_2. But can we also find the other properties behind the shock wave? It turns out that we can, using

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)\,M_1^2}{2+(\gamma-1)\,M_1^2}, \tag{5.3}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right), \tag{5.4}$$

$$\frac{T_2}{T_1} = \frac{h_2}{h_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} = \left(1 + \frac{2\gamma}{\gamma+1}\left(M_1^2-1\right)\right)\left(\frac{2+(\gamma-1)\,M_1^2}{(\gamma+1)\,M_1^2}\right). \tag{5.5}$$

It would also be interesting to know how the total temperature T_t and the total pressure p_t change across the shock wave. Since a shockwave is an adiabatic process we know that $h_1 = h_2$ and thus also $T_{t,1} = T_{t,2}$. Finally, using the relation for entropy we can find that

$$\frac{p_{t,2}}{p_{t,1}} = e^{-\frac{s_2-s_1}{R}}. \tag{5.6}$$

So what can we derive from all the above equations? When passing through a shock wave, the properties of the flow change drastically. The pressure, temperature and density increase, while the total pressure and the Mach number decrease. The total temperature and the enthalpy stay constant.

6 Measuring the Velocity

When an aircraft is flying, it would be nice to know how fast it is going. To find this out, a Pitot tube is used, measuring the total pressure p_t. We also assume that the static pressure p is known.

To find the velocity during a subsonic flight, we can simply use the relation

$$\frac{p_t}{p} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}. \tag{6.1}$$

Solving for M^2 and using $u^2 = M^2a^2$ we find that

$$u^2 = \frac{2a^2}{\gamma-1}\left(\left(\frac{p_t}{p}\right)^{\frac{\gamma-1}{\gamma}} - 1\right). \tag{6.2}$$

So to find the velocity, we also need to know the speed of sound. But if we know that, it's easy to find the velocity.

To find the velocity during a supersonic flight is a bit more difficult, since there is a shock wave. This time the Pitot tube measures the total pressure behind the shock wave $p_{t,2}$. The static pressure that was known is now called p_1. This time we need to use the relation

$$\frac{p_{t,2}}{p_1} = \frac{p_{t,2}}{p_2}\frac{p_2}{p_1} = \left(\frac{(\gamma+1)^2\,M_1^2}{4\gamma M_1^2 - 2\,(\gamma-1)}\right)^{\frac{\gamma}{\gamma-1}}\frac{(1-\gamma)+2\gamma M_1^2}{\gamma+1}. \tag{6.3}$$

This equation is called the **Rayleigh Pitot tube formula**. In its derivation we used the normal shock wave relations for the ratio p_2/p_1. We used the relation for total pressure in an isentropic flow for the ratio $p_{t,2}/p_2$. From this equation the Mach number can be solved. Then only the speed of sound is still needed to find the flight velocity u.