Navier-Stokes Equations

1 Continuity equation

The **continuity equation** is based on conservation of mass. Let's look at a volume ν with surface S, which is fixed in space. The mass flow out of this volume B is equal to the decrease of mass inside the volume C.

The mass flow through a certain area $\mathbf{dS}$ is $\rho \mathbf{V} \cdot \mathbf{dS}$. Since $\mathbf{dS}$ points outward, we're looking at the mass flowing outward. To find the total mass flowing outward, we just integrate over the surface S, to find that

$$B = \iint_S \rho \mathbf{V} \cdot \mathbf{dS}. \tag{1.1}$$

Now let's find C. The mass in a small volume $d\nu$ is $\rho\, d\nu$. The total mass in the volume ν can be found by a triple integral. But we're not looking for the total mass, but for the rate of mass decrease. So we simply take a time derivative of the mass. This gives

$$C = -\frac{\partial}{\partial t} \iiint_\nu \rho\, d\nu. \tag{1.2}$$

Note that the minus is there, because we're looking for the rate of mass decrease. (Not increase!) Using $B = C$ we can find the continuity equation

$$\frac{\partial}{\partial t} \iiint_\nu \rho\, d\nu + \iint_S \rho \mathbf{V} \cdot \mathbf{dS} = 0. \tag{1.3}$$

Since the control volume is fixed, we can pull $\frac{\partial}{\partial t}$ within the integral. And by using Gauss' divergence theorem, we can rewrite this to

$$\iiint_\nu \frac{\partial \rho}{\partial t}\, d\nu + \iiint_\nu \nabla \cdot (\rho \mathbf{V})\, d\nu = \iiint_\nu \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) \right) d\nu = 0. \tag{1.4}$$

Now it may be assumed that, for every small volume $d\nu$ in the volume ν, the integrand is zero:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0. \tag{1.5}$$

Note that in the case of a steady flow $\frac{\partial \rho}{\partial t} = 0$, so also $\nabla \cdot (\rho \mathbf{V}) = 0$. And if the flow is also incompressible, then $\nabla \cdot \mathbf{V} = 0$. The value $\nabla \cdot \mathbf{V}$ occurs relatively often in equations and will be discussed later.

2 Momentum equation

The **momentum equation** is based on the principle "Sum of forces = Time rate of change of momentum". Let's look once more at a fixed volume in space ν with boundary surface S. First we'll examine the forces acting on it. Then we'll examine the change in momentum.

Two types of forces can act on our volume ν. Body forces, such as gravity, and surface forces, such as pressure and shear stress. First let's look at the body forces. Suppose $\mathbf{f}$ represents the net body force per unit mass exerted on the fluid inside ν. On a small volume $d\nu$, the body force is $\rho \mathbf{f}\, d\nu$. So the total body force is

$$\iiint_\nu \rho \mathbf{f}\, d\nu. \tag{2.1}$$

Now let's examine the surface forces. On a small surface $\mathbf{dS}$ acts a pressure p, directed inward. But $\mathbf{dS}$ is directed outward, so the actual force vector caused by the pressure is $-p\,\mathbf{dS}$. The total pressure force therefore is

$$-\iint_S p\,\mathbf{dS}. \tag{2.2}$$

The shear stresses on the volume, caused by viscous forces, may be complicated. So let's just define $\mathbf{F}_{\mathbf{viscous}}$ as the sum of all the viscous stresses. This makes the total force acting on our volume ν

$$\mathbf{F} = \iiint_\nu \rho\mathbf{f}\,d\nu - \iint_S p\,\mathbf{dS} + \mathbf{F}_{\mathbf{viscous}} \tag{2.3}$$

Now let's look at the rate of change of momentum in ν. This consists of two parts. First, particles leave ν, taking momentum with them. From the previous paragraph, we know that the mass flow leaving ν through $\mathbf{dS}$ is $\rho\mathbf{V}\cdot\mathbf{dS}$. So the flow of momentum that leaves ν through $\mathbf{dS}$ is $(\rho\mathbf{V}\cdot\mathbf{dS})\mathbf{V}$. The total momentum leaving ν therefore is

$$\iint_S (\rho\mathbf{V}\cdot\mathbf{dS})\mathbf{V} \tag{2.4}$$

Second, unsteady fluctuations of flow properties inside ν can also cause a change in momentum. The momentum of a small volume $d\nu$ is the mass times the velocity, being $(\rho\,d\nu)\mathbf{V}$. The total momentum of ν can be obtained by integrating. But we don't want the total momentum, but the time rate of change of momentum. So just like in the last paragraph, we put $\frac{\partial}{\partial t}$ in front of it to get

$$\frac{\partial}{\partial t}\iiint_\nu \rho\mathbf{V}\,d\nu \tag{2.5}$$

We now have calculated both the sum of the forces, and the change in momentum. Time to put it all together in one equation

$$\iiint_\nu \rho\mathbf{f}\,d\nu - \iint_S p\,\mathbf{dS} + \mathbf{F}_{\mathbf{viscous}} = \iint_S (\rho\mathbf{V}\cdot\mathbf{dS})\mathbf{V} + \frac{\partial}{\partial t}\iiint_\nu \rho\mathbf{V}\,d\nu \tag{2.6}$$

Just like we did in the previous paragraph, we can use the gradient theorem to bring the entire equation under one integral. Let's define $\Im_{\mathbf{viscous}}$ as the part of $\mathbf{F}_{\mathbf{viscous}}$ acting on a small volume $d\nu$. If we simplify the equation and split it up in components, we find

$$\rho f_x - \frac{\partial p}{\partial x} + \Im_{x_{viscous}} = \frac{\partial(\rho u)}{\partial t} + \nabla\cdot(\rho u\mathbf{V}), \tag{2.7}$$

$$\rho f_y - \frac{\partial p}{\partial y} + \Im_{y_{viscous}} = \frac{\partial(\rho v)}{\partial t} + \nabla\cdot(\rho v\mathbf{V}), \tag{2.8}$$

$$\rho f_z - \frac{\partial p}{\partial z} + \Im_{z_{viscous}} = \frac{\partial(\rho w)}{\partial t} + \nabla\cdot(\rho w\mathbf{V}). \tag{2.9}$$

If the flow is steady ($\frac{\partial}{\partial t} = 0$), inviscid ($\mathbf{F}_{\mathbf{viscous}} = 0$) and if there are no body forces ($\mathbf{f} = 0$), these equations reduce to

$$-\frac{\partial p}{\partial x} = \nabla\cdot(\rho u\mathbf{V}) \tag{2.10}$$

$$-\frac{\partial p}{\partial y} = \nabla\cdot(\rho v\mathbf{V}) \tag{2.11}$$

$$-\frac{\partial p}{\partial z} = \nabla\cdot(\rho w\mathbf{V}) \tag{2.12}$$

If the flow is incompressible (ρ is constant), we have four equations (the momentum equation has three components) and four unknowns, being p, u, v and w. It can be solved. But if ρ is not constant, we need an additional equation.

3 Energy equation

The **energy equation** is based on the principle that energy can be neither created nor destroyed. Let's once more take a fixed volume ν with boundary surface S. We will be looking at the time rate of change of energy. But first we make a few definitions. B_1 is the rate of heat added to ν. B_2 is the rate of work done on ν. B_3 is the rate of change of energy in ν. So all values are rates of changes and therefore have unit J/s. Putting it all together gives something similar to the first law of thermodynamics. The relation between B_1, B_2 and B_3 is

$$B_1 + B_2 = B_3. \tag{3.1}$$

First let's look at B_1. The heat can increase by volumetric heating (for example due to radiation). Let's denote the volumetric rate of heat addition per unit mass be denoted by $\dot{q}[J/kg\,s]$. The heating of a small volume $d\nu$ is $\dot{q}\rho\, d\nu$.

In addition, if the flow is viscous, heat can be transferred across the surface, for example by thermal conduction. This is a complicated thing, so let's just denote the rate of heat addition due to viscous effects by $\dot{Q}_{viscous}$. Now we know that B_1 is

$$B_1 = \iiint_\nu \dot{q}\rho\, d\nu + \dot{Q}_{viscous}. \tag{3.2}$$

Now let's look at B_2. The rate of work done on a body is $\mathbf{F} \cdot \mathbf{V}$. Just like in the previous paragraph, three forces are acting on a small volume $d\nu$. Body forces ($\rho\mathbf{F}\, d\nu$), pressure forces ($-p\,\mathbf{dS}$) and viscous forces. Let's denote the contribution of the friction forces to the work done by $\dot{W}_{viscous}$. Putting it all together gives

$$B_2 = \iiint_\nu \rho(\mathbf{f} \cdot \mathbf{V})\, d\nu - \iint_S p\mathbf{V} \cdot \mathbf{dS} + \dot{W}_{viscous} \tag{3.3}$$

To find B_3, we look at the energy in ν. The internal energy in ν is denoted by e, while the kinetic energy per unit mass if $\frac{V^2}{2}$. The total energy per unit mass is simply $E = e + \frac{V^2}{2}$.

The particles leaving ν through the surface S take energy with them. The mass flow leaving through a surface $\mathbf{dS}$ is still $\rho\mathbf{V} \cdot \mathbf{dS}$. Multiply this by the energy per unit mass gives $\rho E(\mathbf{V} \cdot \mathbf{dS})$, being the rate of energy leaving ν through $\mathbf{dS}$. To find the total rate of energy leaving, simply integrate over the surface S.

In addition, if the flow is unsteady, the energy inside ν can also change due toe transient fluctuations. The energy of a small volume $d\nu$ is $\rho E\, d\nu$. The total energy can be obtained by integrating over the volume ν. But we don't want the total energy, we want the time rate of change of energy. So, just like in the last two paragraphs, we use $\frac{\partial}{\partial t}$. Now we have enough data to find B_3, which is

$$B_3 = \iint_S \rho\left(e + \frac{V^2}{2}\right)\mathbf{V} \cdot \mathbf{dS} + \frac{\partial}{\partial t}\iiint_\nu \rho\left(e + \frac{V^2}{2}\right) d\nu. \tag{3.4}$$

Putting everything together gives us the energy equation

$$\iiint_\nu \rho\dot{q}\, d\nu + \dot{Q}_{viscous} + \iiint_\nu \rho(\mathbf{f}\cdot\mathbf{V})\, d\nu - \iint_S p\mathbf{V}\cdot\mathbf{dS} + \dot{W}_{viscous} = \iint_S \rho\left(e + \frac{V^2}{2}\right)\mathbf{V}\cdot\mathbf{dS} + \frac{\partial}{\partial t}\iiint_\nu \rho\left(e + \frac{V^2}{2}\right) d\nu. \tag{3.5}$$

Just like in the previous paragraphs, we can follow steps to remove the triple integral. Doing this results in

$$\rho\dot{q} + \rho(\mathbf{f} \cdot \mathbf{V}) - \nabla \cdot (p\mathbf{V}) + \dot{Q}'_{viscous} + \dot{W}'_{viscous} = \nabla \cdot \left(\rho\left(e + \frac{V^2}{2}\right)\mathbf{V}\right) + \frac{\partial}{\partial t}\left(\rho\left(e + \frac{V^2}{2}\right)\right), \tag{3.6}$$

where $\dot{Q}'_{viscous}$ and $\dot{W}'_{viscous}$ represent the proper forms of the viscous terms after being put inside the triple integral.

If the flow is steady ($\frac{\partial}{\partial t} = 0$), inviscid ($\dot{Q}_{viscous} = 0$ and $\dot{W}_{viscous} = 0$), adiabatic (no heat addition, $\dot{q} = 0$) and without body forces ($\mathbf{f} = 0$), the energy equation reduces to

$$\nabla \cdot \left(\rho \left(e + \frac{V^2}{2} \right) \mathbf{V} \right) = -\nabla \cdot (p\mathbf{V}). \tag{3.7}$$

4 Equation of state

Now we have five equations, but six unknowns, being p, ρ, u, v, w and e. To solve it, we need more equations. If the gas is perfect, then

$$e = c_v T, \tag{4.1}$$

where c_v is the specific gas constant for constant volume and T is the temperature. But this gives us yet another unknown variable, being the temperature. To complete the system, we can make use of the equation of state

$$p = \rho R T. \tag{4.2}$$

We now have seven unknowns and seven equations, which means the system can be solved.

5 Substantial derivative

Suppose we look at a very small point in space (from a stationary reference frame). The density changes according to $\frac{\partial \rho}{\partial t}$. But now let's look at a very small volume in space (from a co-moving reference frame). The time rate of change of this volume is defined as the **substantial derivative** $\frac{D\rho}{Dt}$. It can be shown that this derivative is given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{V} \cdot \nabla) \qquad \Leftrightarrow \qquad \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\mathbf{V} \cdot \nabla)\, \rho. \tag{5.1}$$

Of course the ρ can be replaced by other variables. The first $\frac{\partial}{\partial t}$ is called the **local derivative** and the second part $(\mathbf{V} \cdot \nabla)$ is called the **convective derivative**.

The substantial derivative can be used to write the Navier-Stokes equations in a simpler form. To do that, we make use of a vector relation, which is rather similar to the chain rule, being

$$\nabla \cdot (\rho \mathbf{V}) = \rho(\nabla \cdot \mathbf{V}) + (\nabla \rho) \cdot \mathbf{V} = \rho \nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla \rho. \tag{5.2}$$

Applying this relation and the substantial derivative to the continuity equation (equation 1.5) gives

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} = 0. \tag{5.3}$$

Using the same tricks, the momentum equation (equation 2.7 to 2.9) can be rewritten as

$$\rho f_x - \frac{\partial p}{\partial x} + \Im_{x_{viscous}} = \rho \frac{Du}{Dt} \tag{5.4}$$

$$\rho f_y - \frac{\partial p}{\partial y} + \Im_{y_{viscous}} = \rho \frac{Dv}{Dt} \tag{5.5}$$

$$\rho f_z - \frac{\partial p}{\partial z} + \Im_{z_{viscous}} = \rho \frac{Dw}{Dt} \tag{5.6}$$

If the flow is steady ($\frac{\partial}{\partial t} = 0$) and inviscid ($\mathbf{F_{viscous}} = 0$), these equations can be simplified even more.

Now let's look at the energy equation (equation 3.6). In the same way as the above equations, it can be rewritten. The outcome is

$$\dot{q}\rho + \rho(\mathbf{f} \cdot \mathbf{V}) - \nabla \cdot (p\mathbf{V}) + \dot{Q}'_{viscous} + \dot{W}'_{viscous} = \rho \frac{D\left(e + \frac{V^2}{2}\right)}{Dt}. \tag{5.7}$$

It is conventional to call the earlier forms of the equations (equations 1.5, 2.7 to 2.9 and 3.6) the **conservation form** (or sometimes the **divergence form**), while the equations of this paragraph are called the **non-conservation form**. In most cases, there is no particular reason to choose one form over the other.

6 Divergence of velocity

The quantity $\nabla \cdot \mathbf{V}$ occurs frequently in equations. Let's consider an amount of air ν from a co-moving reference frame. As the air moves, the volume of ν can change. We will take a look at that change now.

Let's consider a small bit of surface $\mathbf{dS}$ of ν. This surface moves. The change in volume that this piece of surface causes is $\mathbf{V} \cdot \mathbf{dS}$. So the total change in volume per unit time can be found, using an integral over the surface, giving

$$\frac{D\nu}{Dt} = \iint_S \mathbf{V} \cdot \mathbf{dS} = \iiint_\nu (\nabla \cdot \mathbf{V}) d\nu. \tag{6.1}$$

The latter part is known due to the divergence theorem. Note that we have used the substantial derivative $\frac{D\nu}{dt}$ instead of $\frac{d\nu}{dt}$ since we are considering a moving volume of air, instead of air passing through a fixed volume in space.

If the volume ν is small enough, such that $\nabla \cdot \mathbf{V}$ is the same everywhere in ν, then we can find that

$$\nabla \cdot \mathbf{V} = \frac{1}{\nu} \frac{D\nu}{Dt}. \tag{6.2}$$

This equation states that $\nabla \cdot \mathbf{V}$ is the time rate of change of the volume of a moving fluid element per unit volume. This sounds complicated, but an example will illustrate this fact. If $\nabla \cdot \mathbf{V} = -0.8s^{-1}$, then the volume ν will decrease by 80% every second (the minus sign indicates a decrease). If $\nabla \cdot \mathbf{V} = 1s^{-1}$, then the volume ν will double in size every second (that is, as long $\nabla \cdot \mathbf{V}$ remains $1s^{-1}$).