Flow Through Wind Tunnels

To be able to test with supersonic flows, wind tunnels are used. To reach a supersonic flow, they must have a characteristic shape. Why is this? And how does this shape effect the flow? We will try to find that out.

1 Basic Equations

Let's consider a wind tunnel. The flow in it is not entirely one-dimensional. As the cross-section changes, the flow also goes in the y and z-direction. However, if we assume that the cross-section changes only very gradually, then these components are small with respect to the x-direction. We would then approximately have a one-dimensional flow: a so-called **quasi-one-dimensional flow**. In this flow all parameters p, ρ, u and also A only depend on x.

What equations hold for such a flow? From the general continuity, momentum and energy equation we can derive that

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2, \tag{1.1}$$

$$p_1 A_1 + \rho_1 u_1^2 A_1 + \int_{A_1}^{A_2} p\, dA = p_2 A_2 + \rho_2 u_2^2 A_2, \tag{1.2}$$

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}. \tag{1.3}$$

There aren't many surprises in the first and third of these equations. But in the middle one is an integral! This is because the walls of the wind tunnel aren't horizontal. They can thus also exert a pressure force in x-direction on the flow.

Of course having an integral in an equation isn't convenient. To prevent that, we simply consider two points, with an infinitely small distance dx between them. So we would then have $p_2 = p_1 + dp$, $\rho_2 = \rho_1 + d\rho$, and so on. Filling this in, and working it all out, we would get

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0, \tag{1.4}$$

$$dp = -\rho u\, du, \tag{1.5}$$

$$dh + u\, du = 0. \tag{1.6}$$

The middle one of these three equations (the one derived from the momentum equation) is called **Euler's equation**. Using the above equations, we can derive new equations for the flow through wind tunnels, as we will see in the coming paragraph.

2 Area, Velocity and Mach Number

We can extensively rewrite and combine the equations we just found. By doing so, we can derive another important relation, called the **area-velocity relation**. It states that

$$\frac{dA}{A} = \left(M^2 - 1\right) \frac{du}{u}. \tag{2.1}$$

Now what does this equation tell us? First let's suppose that $M < 1$. If the cross-sectional area gets bigger ($dA > 0$), then the velocity decreases ($du < 0$). Also, if the area gets smaller, the velocity increases. This is rather intuitive. However, the counterintuitive part comes when $M > 1$. Now things are exactly opposite. If the area gets bigger, then the velocity also increases. Similarly, if the wind tunnel decreases in size, then the flow also reduces its velocity.

A special case occurs if $M = 1$. If this is true, then we must have $dA = 0$. So a sonic flow can only occur when the cross-section is at a minimum (at a so-called **throat**). Note that the flow properties at this point are the flow properties at sonic conditions, which we denoted with a star (*). So we would have a pressure p^*, a density ρ^* and a flow velocity $u^* = a^*$.

So we have found that the cross-sectional area A and the Mach number M are linked. But how? To find that out, we can derive that

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left(\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right)^{\frac{\gamma+1}{\gamma-1}}, \tag{2.2}$$

where A^* is the cross-sectional area at the throat. (Note that if we fill in $M = 1$ we would get $A = A^*$.) This important equation is called the **area-Mach number relation**. It shows that the Mach number only depends on the ratio A/A^*.

3 Flow in a Nozzle

Let's examine a nozzle. We can consider three parts in it. To the left is a reservoir of air. At this point the cross-sectional area A is very big. The velocity is therefore very low. The pressure and temperature at this point are thus equal to the total pressure p_t and the total temperature T_t.

In the middle of the nozzle is the throat. To the right of that, the tunnel gets wider again. Eventually there is an exit, with exit pressure p_e and exit temperature T_e.

Flow doesn't go through the nozzle spontaneously. It flows because $p_e < p_t$. This pressure difference causes the air to move. However, the flow doesn't always reach supersonic velocities. To check how the flow behaves, we need to examine the ratio p_e/p_t. While doing that, we can consider 6 stages.

In the first stage, the ratio $p_{e,1}/p_t \approx 1$. This causes the flow to move, but only slowly. Not much special is going on. In the second stage, the ratio $p_{e,2}/p_t$ becomes smaller. However, the flow remains subsonic. In stage three, the ratio $p_{e,3}/p_t$ is sufficiently small to cause a sonic flow in the throat. So at the throat finally $M = 1$. However, after the throat the flow becomes subsonic again.

Now what happens if we decrease the exit pressure even further? We then reach stage four. In this stage, the flow becomes supersonic after the throat. However, the pressure difference isn't big enough to continue this supersonic flow. So a normal shock wave appears, slowing the flow down to subsonic velocities. To know where the shock wave appears, you have to look at the pressure. The pressure drop in the normal shock wave should be such that, at the exit, the exit pressure $p_{e,4}$ is reached.

If we decrease the exit pressure further, the position of the normal shock wave changes. In fact, it moves to the right. This continues until we reach stage 5. In stage 5 the normal shock wave is at the exit of the nozzle.

If we decrease the exit pressure just a little bit further, we reach stage 6. A supersonic flow now exits the nozzle. In this case the exit pressure is always a fixed value $p_{e,6}$. However, now the **back pressure** p_B is also important. This is the pressure behind the nozzle. (Previously the back pressure p_B was equal to the exit pressure p_e. Now this is not the case.) We can now consider three cases:

- If $p_B > p_{e,6}$, the flow has expanded too much (it is **overexpanded**) and will decrease in size once it exits the nozzle. This causes oblique shock waves.
- If $p_B < p_{e,6}$, the flow hasn't expanded enough (it is **underexpanded**) and will increase in size once it exits the nozzle. Due to this, expansion waves will occur.
- If $p_B = p_{e,6}$, the flow will just exit the nozzle without any waves.

During stages 3 to 6, an important phenomenon occurs. In all these stages, we have $M = 1$ at the throat. From this follows that the pressure in the throat is always

$$p^* = \frac{p_t}{\left(1 + \frac{\gamma-1}{2}\right)^{\frac{\gamma}{\gamma-1}}} = 0.528 p_t. \tag{3.1}$$

From this we can derive that the mass flow in the throat is always the same during stages 3 to 6. So although the exit pressure (or back pressure) decreases, the mass flow through the nozzle stays the same. This effect is called **choked flow**.

4 Wind Tunnels

Suppose we have a model of an airplane, which we want to test at supersonic velocities. What kind of wind tunnel do we need? We can simply take a nozzle, having only one throat. If we do this, we can get supersonic velocities. However, a huge pressure ratio p_t/p_e will be needed. This means expensive equipment, which is of course undesirable.

The solution lies in a **diffuser**. A diffuser slows the flow down, back to subsonic velocities. During this process, the pressure increases. So in a wind tunnel we would first have a nozzle, then our test model, and finally a diffuser. To the left of the nozzle is the high pressure p_t. At the test model is a low pressure, but a high velocity. Finally, after the diffuser, there is a low velocity, but a more or less high pressure p_e. Although still $p_e < p_t$, the ratio p_t/p_e is much smaller than normal. This therefore makes supersonic wind tunnels feasible.

Let's take a closer look at this diffuser. A diffuser has a similar shape as a nozzle: it has a throat. However, this time there is a supersonic flow ($M > 1$) to the left of the throat, and a subsonic flow to the right. Once more, we have $M = 1$ at the throat. After the throat will be a subsonic flow ($M < 1$). Ideally, this would occur isentropically, without any shock waves. In reality, there are viscous effects near the edges of the diffuser. These viscous effects eventually cause shock waves.

When designing a wind tunnel, we would like to know how big the cross-sectional area of the diffuser throat should be. In the wind tunnel we will be having two throats: one in the nozzle (with cross-sectional area $A_{t,1}$) and one in the diffuser (with area $A_{t,2}$). (Note that the subscript t now stands for throat; not total.) These areas relate to each other, according to

$$\frac{A_{t,2}}{A_{t,1}} = \frac{p_{t,1}}{p_{t,2}}. \tag{4.1}$$

In an ideal (isentropic) situation we have $p_{t,1} = p_{t,2}$, so also $A_{t,1} = A_{t,2}$. In reality, however, there are viscous effects. Due to this we have $p_{t,1} > p_{t,2}$ and thus also $A_{t,2} > A_{t,1}$. So the diffuser throat should always be bigger than the nozzle throat.

Now what happens if $A_{t,2}$ is too small? In this case the diffuser will **choke**. It can't handle the mass flow. This causes shock waves in the test section. This can ultimately lead to an entirely subsonic test section. In such a case, the wind tunnel is said to be **unstarted**.