Enthalpy, specific heat and isentropic flows

1 Definitions

h = Enthalpy (J/kg)

e = The specific internal energy in a gas (J/kg) (**Note:** This is in some books given the name u)

R = Gas constant (Value is: $287 J/(kgK)$) ($J/(kgK)$)

T = The temperature of a certain amount of gas (K)

p = Pressure ($Pa = N/m^2$)

v = Specific volume (defined as $\frac{1}{\rho}$) (m^3/kg)

δq = The added heat into a certain amount of gas (J)

c = Specific heat ($J/(kgK)$)

c_v = Specific heat at constant volume (For air: $c_v = 717 J/(kgK)$) ($J/(kgK)$)

c_p = Specific heat at constant pressure (For air: $c_p = 1004 J/(kgK)$) ($J/(kgK)$)

γ = Ratio of specific heats (Value for normal air is: 1.4) (dimensionless)

ρ = Density (kg/m^3)

2 Enthalpy

The enthalpy h is defined as follows:

$$h = e + RT = e + pv \tag{2.1}$$

To differentiate that, the chain rule must be applied, since neither p or v are constant. So:

$$dh = de + pdv + vdp \tag{2.2}$$

3 Specific heat

Note: Almost all of the following formulas apply in general, as long as the gas is a perfect gas.

From basic thermodynamics, it can be derived that:

$$\delta q = de + pdv \tag{3.1}$$

When adding an amount of heat to 1 kg of a gas, the temperature of the gas increases. The relation is (per definition):

$$c = \frac{\delta q}{dT} \tag{3.2}$$

The addition of heat can be done in multiple ways. If the volume is kept constant (thus $dv = 0$), the following formulas apply:

$$c_v = \frac{\delta q}{dT} \tag{3.3}$$

$$c_v dT = \delta q = de + pdv = de$$

$$e = c_v T \tag{3.4}$$

But if the pressure is kept constant (thus $dp = 0$), it is slightly more difficult. For that we will use the enthalpy. The following formulas now apply:

$$c_p = \frac{\delta q}{dT} \tag{3.5}$$

$$c_p dT = \delta q = de + pdv = dh - vdp = dh$$

$$h = c_p T \tag{3.6}$$

Between the two specific heat constants are interesting relationships. Their difference can be derived:

$$c_p - c_v = \frac{c_p T - c_v T}{T} = \frac{h - e}{T} = \frac{RT}{T} = R$$

And their ratio is (per definition):

$$\gamma = \frac{c_p}{c_v} \tag{3.7}$$

4 Specific heat ratio in isentropic processes

An isentropic process is both an adiabatic process ($\delta q = 0$) as a reversible process (no friction forces). Suppose there is an isentropic process present. It can in that case be shown that:

$$\frac{dp}{p} = -\frac{c_p}{c_v}\frac{dv}{v} = -\gamma\frac{dv}{v}$$

Integrating that equation and working out the results would give us:

$$\ln\frac{p_2}{p_1} = -\gamma \ln\frac{v_2}{v_1}$$

And since $v = \frac{1}{\rho}$, it can be derived that:

$$\frac{p_1}{p_2} = \left(\frac{\rho_1}{\rho_2}\right)^\gamma$$

Using the equation of state ($\frac{p}{\rho} = RT$) it can also be derived that:

$$\frac{p_1}{p_2} = \left(\frac{T_1}{T_2}\right)^{\frac{\gamma}{\gamma-1}}$$

And by combining these two formulas, one can also derive the following formula:

$$\frac{p_1}{p_2} = \left(\frac{\rho_1}{\rho_2}\right)^\gamma = \left(\frac{T_1}{T_2}\right)^{\frac{\gamma}{\gamma-1}} \tag{4.1}$$

These important relations are called the isentropic flow relations, and are only relevant to compressible flow.

5 Energy equation

We still assume an isentropic process (so $\delta q = 0$). Therefore $\delta q = dh - vdp = 0$, and since the Euler equation says that $dp = -\rho V dV$ we know that $dh + v\rho V dV = 0$. However, $v = \frac{1}{\rho}$ so also $dh + V dV = 0$. Integrating, and working out the results of it, would give:

$$h_1 + \frac{1}{2}V_1^2 = h_2 + \frac{1}{2}V_2^2 \tag{5.1}$$

And since $h = c_p T$, also the following equation is true:

$$c_p T_1 + \frac{1}{2}V_1^2 = c_p T_2 + \frac{1}{2}V_2^2 \tag{5.2}$$

This equation is called the energy equation.