Elementary Flows

1 Uniform flows

A **uniform flow** (oriented in the positive x-direction) is a flow with velocity components $u = V_\infty$ and $v = 0$ everywhere. Such a flow is irrotational. It therefore has a velocity potential ϕ, which can be shown to be

$$\phi = V_\infty x. \tag{1.1}$$

Also the stream function can be determined to be

$$\psi = V_\infty y. \tag{1.2}$$

Note that these two functions both satisfy Laplace's equation.

In a 3-dimensional world, the velocity potential is the same, and therefore $w = 0$.

2 Source flows

A **source flow** is a flow where all the streamlines are straight lines emanating from a central point O, and where the velocity varies inversely with the distance from O. In formula this is

$$V_r = \frac{\Lambda}{2\pi r}, \qquad V_\theta = 0, \tag{2.1}$$

where Λ is the source stength. Such a flow is incompressible (except at point O itself) and irrotational. If $\Lambda > 0$, we are dealing with a source flow. If $\Lambda < 0$, we are looking at a so-called **sink flow**, where the velocity vectors point inward.

The velocity potential of the flow can be found using the above velocity relations. The result will be

$$\phi = \frac{\Lambda}{2\pi} \ln r. \tag{2.2}$$

Note that this function is not defined for $r = 0$, since the flow is not incompressible there. Identically, the stream function can be shown to be

$$\psi = \frac{\Lambda}{2\pi}\theta. \tag{2.3}$$

Now let's look at 3-dimensional sources. 3-Dimensional source flows are similar to 2-dimensional ones. Let's define λ as the volume flow originating from the source. The velocity is now, in spherical coordinates,

$$V_r = \frac{\lambda}{4\pi r^2}, \qquad V_\theta = 0, \qquad V_\phi = 0. \tag{2.4}$$

The stream function is not defined for 3-dimensional situations. The velocity potential is

$$\phi = -\frac{\lambda}{4\pi r}. \tag{2.5}$$

3 Doublets

Suppose we have a source of strength Λ at coordinates $(-\frac{1}{2}l, 0)$ and a source of strength $-\Lambda$ (thus being a sink) at coordinates $(\frac{1}{2}l, 0)$. If $l \to 0$, we obtain a flow pattern called a **doublet**. The **strength** of the doublet is defined as $\kappa = l\Lambda$. So as $l \to 0$ also $\Lambda \to \infty$. The velocity potential now is

$$\phi = \frac{\kappa}{2\pi}\frac{\cos\theta}{r}. \tag{3.1}$$

Also, the stream function is

$$\psi = -\frac{\kappa}{2\pi}\frac{\sin\theta}{r}. \tag{3.2}$$

The streamlines of a doublet are therefore given by

$$\psi = c \qquad \Rightarrow \qquad r = -\frac{\kappa}{2\pi c}\sin\theta. \tag{3.3}$$

It can mathematically be shown that these are circles with diameter $d = \frac{\kappa}{2\pi c}$ and with their centers positioned at coordinates $(0, \pm\frac{1}{2}d)$.

Now let's look at 3-dimensional doublets. Just like in a 2-dimensional doublet, a 3-dimensional doublet has a 3-dimensional source and sink at a very small distance from each other. The 3-dimensional doublet strength is defined as $\mu = \Lambda l$. The velocity potential then is

$$\phi = -\frac{\mu}{4\pi}\frac{\cos\theta}{r^2}. \tag{3.4}$$

4 Vortex flows

A **vortex flow** is a flow in which the stream lines form concentric circles about a given point. Such a flow is described by

$$V_r = 0, \qquad V_\theta = -\frac{\Gamma}{2\pi r}, \tag{4.1}$$

where Γ is the circulation. In this case Γ is also called the **strength** of the vortex. A positive strength corresponds to a clockwise vortex, while a counterclockwise vortex indicates a negative strength.

Vortex flow is irrotational everywhere except at $r = 0$, where the vorticity is infinite. The velocity potential is given by

$$\phi = -\frac{\Gamma}{2\pi}\theta. \tag{4.2}$$

Also, the stream function is

$$\psi = \frac{\Gamma}{2\pi}\ln r. \tag{4.3}$$

There are no 3-dimensional vortex flows. The only way in which three-dimensional vortex flows can occur is if multiple 2-dimensional vortex flows are stacked on top of each other. This is then, in fact, still a 2-dimensional problem and can be solved with the above equations.

5 Elementary flow overview

Flow type	**Velocity**	**Velocity potential**	**Stream function**
Uniform flow in x-direction	$u = V_\infty$ $v = 0$	$\phi = V_\infty x$	$\psi = V_\infty y$
Source/Sink	$V_r = \frac{\Lambda}{2\pi r}$ $V_\theta = 0$	$\phi = \frac{\Lambda}{2\pi}\ln r$	$\psi = \frac{\Lambda}{2\pi}\theta$
Doublet	$V_r = -\frac{\kappa}{2\pi}\frac{\cos\theta}{r^2}$ $V_\theta = -\frac{\kappa}{2\pi}\frac{\sin\theta}{r^2}$	$\phi = \frac{\kappa}{2\pi}\frac{\cos\theta}{r}$	$\psi = -\frac{\kappa}{2\pi}\frac{\sin\theta}{r}$
Vortex	$V_r = 0$ $V_\theta = -\frac{\Gamma}{2\pi r}$	$-\frac{\Gamma}{2\pi}\theta$	$\psi = \frac{\Gamma}{2\pi}\ln r$