Dimensional Analysis

1 Definitions

L = Lift (N)

V = Air velocity (m/s)

ρ = Density (kg/m^3)

S = Wing surface (m^2)

a = Speed of sound (m/s)

μ = Viscosity ($kg/(m\,s)$)

2 Introduction to Dimensional Analysis

Suppose the value of a certain variable x_1 depends only on the variables $x_2, x_3, \ldots, x_k$. So $x_1 = f(x_2, x_3, \ldots, x_k)$. Now assume that the function f is a function, such that $x_1 = \phi\, x_2^{p_2}\, x_3^{p_3}\, \ldots\, x_k^{p_k}$, where ϕ and p_i are constant coefficients for every i, then the units of both sides must be equal. This is the assumption dimensional analysis is based on.

Suppose k is the amount of variables you have. These variables all have certain units. However, there is often a relation between units. For example, $1N = 1\frac{kg\,m}{s^2}$ according to Newton's second law. All the variables have to have their units set back to certain more or less basic units. It is normal to choose M, L and T for that, for mass, length and time respectively. Suppose r is the amount of basic units needed. Usually $r = 3$, but sometimes $r = 2$ or $r = 4$. In rare cases r can even get up to 7, but this is rather unlikely to occur. In table 1 certain variables have been expressed in their basic components.

Units	F (N)	p (Pa)	μ ($\frac{kg}{m\,s}$)	ρ ($\frac{kg}{m^3}$)	V ($\frac{m}{s}$)
Basic Units	MLT^{-2}	$ML^{-1}T^{-2}$	$ML^{-1}T^{-1}$	ML^{-3}	LT^{-1}

Table 1: Variables in their "basic" components.

3 Buckingham Pi-Theorem generally explained

To use the Buckingham Pi-Theorem, the following steps should be taken. These steps might seem vague, but comparing the general steps with the example of the next paragraph might offer some clarity at times.

- You first need to write down all the depending variables $x_1, x_2, \ldots, x_k$ and their corresponding units resolved into the basic units.

- Now pick r repeating variables (for the definition of r, see previous paragraph). It is often wise to pick variables that often appear in well-known dimensionless coefficients. The velocity V is a good example, and also ρ and S are often used in the case of aerodynamics. Dimensional analysis with other variables will work just fine as well, but the outcome may be slightly different. You should not pick x_1 (the variable for which you want to find a function) though. For simplicity we will assume that $r = 3$ (if it's not, the steps are still the same). So let's suppose you've picked x_2, x_3 and x_4 as repeating variables.

- Now do the following steps $k - r$ (so for us this would be $k - 3$) times:
  - Suppose this is the i'th time we've performed these steps. Pick one of the non-repeating variables (one you haven't chosen yet). If $i = 1$ (so if this is the first time you're performing these steps) you should pick x_1. Let's assume you've picked x_j. Now write down the formula:

$$\Pi_i = x_j x_2^b x_3^c x_4^d \tag{3.1}$$

    b, c and d are just dimensionless coefficients. We have not taken a in this case, because in the example of the next paragraph a is already used for the speed of sound, and 2 different meanings of a is rather confusing.
  - Then assume that the dimensions of both side of the equation are equal. Π_i is dimensionless. Suppose x_n for every n has as units $M^{m_n} L^{l_n} T^{t_n}$, then write down the following equation (which must be true):

$$M^0 L^0 T^0 = M^{m_j} L^{l_j} T^{t_j} \left(M^{m_2} L^{l_2} T^{t_2}\right)^b \left(M^{m_3} L^{l_3} T^{t_3}\right)^c \left(M^{z_4} L^{z_4} T^{z_4}\right)^d \tag{3.2}$$

  - Now write down the following r equations (in our case 3):

$$\begin{aligned} 0 &= m_j + bm_2 + cm_3 + dm_4 \\ 0 &= l_j + bl_2 + cl_3 + dl_4 \\ 0 &= t_j + bt_2 + ct_3 + dt_4 \end{aligned} \tag{3.3}$$

  - Solve these r equations for a, b and c (if $r = 4$ you should solve it for a, b, c and d of course). Then write down the following:

$$\Pi_i = x_j x_2^b x_3^c x_4^d \tag{3.4}$$

    But instead of writing down b, c and d you write down the values of it. Now you've found one of the Π-terms. To find the other Π-terms just repeat the previous steps.
- When you've found all of the $k - r$ Π-terms, just write down the following:

$$\Pi_1 = f(\Pi_2, \Pi_3, \ldots, \Pi_{k-r}) \tag{3.5}$$

  But instead of writing the Π's, fill in the "values" you've found in the previous steps. Then the dimensional analysis is completed. Of course it is wise to check whether all the Π-terms are dimensionless.

4 Applying the Buckingham Pi-Theorem

To show how to apply the Buckingham Pi-Theorem, we use an example. Let's use Dimensional Analysis to derive a formula for the lift L, depending on the variables V, ρ, S, a and μ. These are all our variables. There are 6 variables, so $k = 6$. In table 2 the basic units of these variables have been written down.

Variable	L	V	ρ	S	a	μ
Basic Units	MLT^{-2}	LT^{-1}	ML^{-3}	L^2	LT^{-1}	$ML^{-1}T^{-1}$

Table 2: Writing down the variables for the example expressed in basic units.

We now need to choose repeating variables. Note that we have 3 basic units (and we can't simplify it any further), so $r = 3$. We now also know that we need $k - r = 6 - 3 = 3$ Π-terms. Let's choose V, ρ and S as the repeating variables. Now we should find the Π-terms:

1. The first Π-term. Since we want to find a function for L, and since this is the first Π-term, we should pick L as non-repeating variable. Now we should write down the following:

$$\Pi_1 = LV^b\rho^c S^d \tag{4.1}$$

Now we write this in the basic units:

$$M^0L^0T^0 = MLT^{-2}\left(LT^{-1}\right)^b\left(ML^{-3}\right)^c\left(L^2\right)^d \tag{4.2}$$

And we write the equations we need to solve, for M, L and T respectively:

$$\begin{aligned} 0 &= 1 + 0 + c + 0 \\ 0 &= 1 + b - 3c + 2d \\ 0 &= -2 - b + 0 + 0 \end{aligned} \tag{4.3}$$

Solving this gives $b = -2$, $c = -1$ and $d = -1$. Now we have found our first Π-term, so we should write it down according to equation 4.1:

$$\Pi_1 = LV^{-2}\rho^{-1}S^{-1} = \frac{L}{V^2\rho S} \tag{4.4}$$

2. The second Π-term. Now we pick a as non-repeating variable, and perform the steps identically:

$$\Pi_2 = aV^b\rho^c S^d \tag{4.5}$$

$$M^0L^0T^0 = LT^{-1}\left(LT^{-1}\right)^b\left(ML^{-3}\right)^c\left(L^2\right)^d \tag{4.6}$$

$$\begin{aligned} 0 &= 0 + 0 + c + 0 \\ 0 &= 1 + b - 3c + 2d \\ 0 &= -1 - b + 0 + 0 \end{aligned} \tag{4.7}$$

Solving gives $b = -1$, $c = 0$ and $d = 0$. So:

$$\Pi_2 = \frac{a}{V} \tag{4.8}$$

3. The third Π-term. Only one non-repeating variables remains. So we pick μ, and do the same steps once more.

$$\Pi_3 = \mu V^b\rho^c S^d \tag{4.9}$$

$$ML^{-1}T^{-1} = LT^{-1}\left(LT^{-1}\right)^b\left(ML^{-3}\right)^c\left(L^2\right)^d \tag{4.10}$$

$$\begin{aligned} 0 &= 1 + 0 + c + 0 \\ 0 &= -1 + b - 3c + 2d \\ 0 &= -1 - b + 0 + 0 \end{aligned} \tag{4.11}$$

Solving gives $b = -1$, $c = -1$ and $d = -\frac{1}{2}$. So:

$$\Pi_3 = \frac{\mu}{V\rho\sqrt{S}} \tag{4.12}$$

All the Π-terms have been written down, so we can write down the function:

$$\Pi_1 = f(\Pi_2, \Pi_3) \tag{4.13}$$

Filling in the Π-terms gives:

$$\frac{L}{V^2\rho S} = f\left(\frac{a}{V}, \frac{\mu}{V\rho\sqrt{S}}\right) \tag{4.14}$$

This is where the Buckingham Pi-Theorem usually ends. However, in this case we can derive a bit more data. Note that Π_2 is 1 over the Mach number, and Π_3 is 1 over the Reynolds number (since $\sqrt{S}$ is a length). So also the following is true:

$$\frac{L}{V^2 \rho S} = f(M, Re) = \phi M^e Re^f \tag{4.15}$$

Where ϕ, e and f are dimensionless constants. If we define $\frac{C_L}{2} = \phi M^e Re^f$, then:

$$L = C_L \frac{1}{2} \rho V^2 S \tag{4.16}$$

And this is a formula that should look familiar to you.

5 Different method of using Dimensional Analysis

Next to the Buckingham Pi-Theorem there is another method of using dimensional analysis, which is rather similar. This method is often faster, but there are a few cases where you should be very careful not to make assumptions you're not allowed to make. So even though the Buckingham Pi-Theorem is often a bit more work, it is a safer method. And if you feel you're having trouble understanding the Buckingham Pi-Theorem, the following 2 paragraphs might just confuse you and may not be worth reading.

Using this method goes as follows. First write down the following equation:

$$x_1 = \phi \, x_2^{p_2} \, x_3^{p_3} \, \ldots \, x_k^{p_k} \tag{5.1}$$

Where ϕ and p_i (for every i) are just dimensionless coefficients. Now write this down in units, more or less identical as in the paragraph introducing the Buckingham Pi-Theorem. For simplicity we will once more assume that $r = 3$, and that the basic units are M, L and T.

$$M^{m_1} L^{l_1} T^{t_1} = \phi \left(M^{m_2} L^{l_2} T^{t_2}\right)^{p_2} \left(M^{m_3} L^{l_3} T^{t_3}\right)^{p_3} \ldots \left(M^{m_k} L^{l_k} T^{t_k}\right)^{p_k} \tag{5.2}$$

Now write the r corresponding coefficient equations:

$$\begin{aligned} m_1 &= p_2 m_2 + p_3 m_3 + \ldots + p_k m_k \\ l_1 &= p_2 t_2 + p_3 l_3 + \ldots + p_k l_k \\ t_1 &= p_2 l_2 + p_3 t_3 + \ldots + p_k t_k \end{aligned} \tag{5.3}$$

Note that the only unknowns in these equations are the p-coefficients. Normally we would choose x_2, x_3 and x_4 as repeating variables. However, there are no repeating variables in this method, but instead of choosing x_2, x_3 and x_4 as repeating variables, we just solve the previous 3 equations for p_2, p_3 and p_4, thus expressing them in the other p-coefficients. We then fill those values in, in equation 5.1. Then we simplify the equation we get, by bonding all the variables to the power p_i together in brackets for every i (except for the i you have chosen as "repeating"). This probably seems vague, but it might become clearer when you read the example of the next paragraph. But when this "bonding" is done, the equation found should be the same as gotten with the Buckingham-Pi Theorem, as long as the same "repeating" variables were chosen.

6 Example of the second method

We will now demonstrate the second method of using dimensional analysis. We will do this by once more deriving a formula for the lift L. According to the method described in the previous paragraph, and according to table 2, we can write down the following equation:

$$L = \phi V^b \rho^c S^d a^e \mu^f \tag{6.1}$$

In units this is:

$$MLT^{-2} = \left(LT^{-1}\right)^b \left(ML^{-3}\right)^c \left(L^2\right)^d \left(LT^{-1}\right)^e \left(ML^{-1}T^{-1}\right)^f \tag{6.2}$$

The three equations we get are:

$$\begin{aligned} 1 &= 0 + c + 0 + 0 + f \\ 1 &= b - 3c + 2d + e - f \\ -2 &= -b - e - f \end{aligned} \tag{6.3}$$

Normally we would choose V, ρ and S as repeating variables. Therefore we now solve these equations for b, c and d. This results in $b = 2 - e - f$, $c = 1 - f$ and $d = 1 - \frac{f}{2}$. So now we can write down an equation. And according to the described method we should bond variables with the same power in brackets as follows:

$$L = \phi V^{2-e-f} \rho^{1-f} S^{1-\frac{f}{2}} a^e \mu^f = \phi \frac{V^2}{V^e V^f} \frac{\rho}{\rho^f} \frac{S}{S^{\frac{f}{2}}} a^e \mu^f = \phi V^2 \rho S \left(\frac{a}{V}\right)^e \left(\frac{\mu}{V\rho\sqrt{S}}\right)^f \tag{6.4}$$

This method has now been completed. Note that the derived equation is equal to equation 4.15, and that the formula can therefore still be changed in an identical way as was performed after equation 4.15. However, that is not part of this method, and we will leave it to this.