Continuity equation and Bernoulli's equation

1 Definitions

$\dot{m}$ = Mass flow (kg/s)
ρ = Air density (kg/m^3)
A = Area (m^2)
V = Speed (m/s)
p = Pressure ($Pa = N/m^2$)
h = Height (m)
S = Surface (m^2)
r = Radius of the curvature (m)
v = Volume (m^3)
V_a = Velocity difference (m/s)
S_d = Actuator disk surface area (m^2)
T = Thrust (N)
P = Power ($W = J/s$)
η = Efficiency (dimensionless)

2 Continuity equation

The continuity equation states that in a tube, the following must be true:

$$\dot{m}_{in} = \dot{m}_{out}$$

This equation can be rewritten to:

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \tag{2.1}$$

Or for incompressible flows (where $\rho_1 = \rho_2$):

$$A_1 V_1 = A_2 V_2 \tag{2.2}$$

3 Bernoulli's equation

The Euler equation, when gravity forces and viscosity are neglected, is:

$$dp = -\rho V dV \tag{3.1}$$

This formula is also valid for compressible flows. Integration for 2 points along a streamline gives:

$$(p_2 - p_1) + \rho\left(\frac{1}{2}V_2^2 - \frac{1}{2}V_1^2\right) = 0$$

This can be easily transformed to the Bernoulli equation, which says that the total pressure is constant along a streamline:

$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2 \tag{3.2}$$

However, if gravity forces are included, the following variant of the Bernoulli equation can be derived:

$$p_1 + \frac{1}{2}\rho V_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g h_2 \tag{3.3}$$

But do remember that the Bernoulli equation is only valid for inviscid (frictionless) incompressible flows.

4 Bernoulli's equation of curved flow

When looking at an infinitely small part of air, there is a pressure p working on one side of it, and a pressure $p + dp$ working on the other side. This pressure difference causes the flow to bend. If S is the surface and dr is the length of the part of air, then the resultant force on the part of air is:

$$F = ((p + dp) - p) \cdot S = dp \cdot S = \frac{dp}{dr} \cdot S \cdot dr = \frac{dp}{dr} v$$

Where v is the volume of the air. However, when performing a circular motion, the resultant force is:

$$F = \frac{mV^2}{r} = \frac{\rho V^2}{r} v$$

Combining these data, we find:

$$\frac{dp}{dr} = \frac{\rho V^2}{r} \tag{4.1}$$

And therefore the following formula remains constant along a curving flow:

$$\int_{p_1}^{p_2} dp = \int_{r_1}^{r_2} \frac{\rho V^2}{r} dr \tag{4.2}$$

And this is the exact reason why concave shapes have a lower pressure in flows, and convex areas have a higher pressure.

5 Actuator disk thrust

Suppose we have a propeller, blowing air from left to right. Let's call point 0 a point infinitely far to the left (undisturbed flow), point 1 just to the left of the propeller (infinitely close to it), point 2 identical to point 1, but then on the right, and point 3 identical to point 0, but also on the right. In every point n, the airflow has a pressure p_n, a velocity V_n and an area S_n. Since point 1 and 2 are infinitely close to each other, the air velocity in both points is equal, so $V_1 = V_2$. Since point 0 and point 3 are both in the undisturbed flow, the pressure in those points is equal, so $p_0 = p_3$. Let's define $V_{a_1} = V_1 - V_0$, $V_{a_2} = V_3 - V_2$ and $V_{a_t} = V_3 - V_0$, so $V_{a_t} = V_{a_1} + V_{a_2}$. From the Bernoulli equation, the following equations can be derived:

$$p_0 + \frac{1}{2}\rho V_0^2 = p_1 + \frac{1}{2}\rho (V_0 + V_{a_1})^2$$

$$p_2 + \frac{1}{2}\rho (V_0 + V_{a_1})^2 = p_0 + \frac{1}{2}\rho (V_0 + V_{a_t})^2$$

Note that it is not allowed to use bernoulli between point 1 and point 2, since energy is added to the flow there. Combining these two equations, we find the pressure difference:

$$p_2 - p_1 = \rho V_{a_t}(V_0 + \frac{1}{2}V_{a_t})$$

If S_d is the surface area of the actuator disk, then the thrust is:

$$T = S_d(p_2 - p_1)$$

Combining this equation with the previous one gives:

$$T = \rho S_d(V_0 + \frac{1}{2}V_{a_t})V_{a_t}$$

However, the thrust is also equal to the mass flow times the change in speed. The mass flow is $\rho S_d(V_0 + V_{a_1})$, and the change in speed is simply equal to V_{a_t}. So:

$$T = \rho S_d(V_0 + V_{a_1})V_{a_t} \tag{5.1}$$

Combining this equation with the previous one, we find another important conclusion:

$$V_{a_1} = V_{a_2} = \frac{1}{2}V_{a_t} \tag{5.2}$$

6 Actuator disk efficiency

Now let's look at the power and the efficiency of the actuator disk. The power input is equal to the change in kinetic energy of the air per unit time:

$$P_{in} = \frac{1}{2}\rho S_d(V_0 + V_{a_1})((V_0 + V_{a_3})^2 - V_0^2) = \rho S_d(V_0 + V_{a_1})^2 V_{a_t}$$

The power that really is outputted, is simply equal to the force times the velocity of the airplane:

$$P_{out} = TV_0 = \rho S_d(V_0 + V_{a_1})V_{a_t}V_0$$

Combing these two equations, we can find the propeller efficiency:

$$\eta = \frac{P_{out}}{P_{in}} = \frac{TV_0}{P_{in}} = \frac{\rho S_d(V_0 + V_{a_1})V_{a_t}V_0}{\rho S_d(V_0 + V_{a_1})^2 V_{a_t}}$$

Simplifying this gives:

$$\eta = \frac{V_0}{V_0 + V_{a_1}} = \frac{1}{1 + \frac{V_{a_1}}{V_0}} \tag{6.1}$$