Coefficients and supersonic flows

1 Definitions

R = Resultant force that acts on the wing (N)
N = Component of R acting perpendicular to the chord line (N)
A = Component of R acting tangential to the chord line (N)
L = Lift - component of R acting perpendicular to the relative wind (N)
D = Drag - component of R acting parallel to the relative wind (N)
α = Angle of attack (usually the angle between the chord line and the relative wind) (deg)
M = Mach number (dimensionless)
Re = Reynolds number (dimensionless)
c = Chord - the length of the chord line (line between the leading edge and the trailing edge) (m)
c_l = Lift coefficient for infinite wings (dimensionless)
ρ = Air density (kg/m^3)
V = Air velocity (m/s)
S = Wing area (m^2)
q = Dynamic air pressure, defined as $q = \frac{1}{2}\rho V^2$ ($Pa = N/m^2$)
c_d = Drag coefficient for infinite wings (dimensionless)
c_m = Moment coefficient for infinite wings (dimensionless)
C_p = Pressure coefficient (dimensionless)
$C_{p,l}$ = Pressure coefficient on the lower side of the wing (dimensionless)
$C_{p,u}$ = Pressure coefficient on the upper side of the wing (dimensionless)
c_n = Normal force coefficient for infinite wings (dimensionless)
c_a = Axial force coefficient for infinite wings (dimensionless)
M_{cr} = Critical Mach number - Mach number at which shock waves start occurring (dimensionless)
μ = Mach angle - Angle between direction of flight and shock wave boundary (rad/deg)

2 Airfoil nomenclature

During flight, a resultant force R acts on the wing. This force can be resolved into two forces in multiple ways. R can be resolved in N and A, where N is the component perpendicular (normal) to the chord, and A is the component tangential (axial) to the chord. However, R can also be resolved in a component L (lift) perpendicular to the relative wind, and a component D (drag) parallel to the relative wind. These forces have the following relation:

$$L = N\cos\alpha - A\sin\alpha \tag{2.1}$$

$$D = N\sin\alpha + A\cos\alpha \tag{2.2}$$

3 Applying dimensional analysis to infinite wings

Using dimensional analysis it can be found that:

$$L = Z\rho_\infty V_\infty^2 S \left(\frac{1}{M_\infty}\right)^e \left(\frac{1}{Re_c}\right)^f$$

Where Z is a constant (as long as α remains constant, because if α changes, also Z changes). So if we define the lift coefficient c_l such that:

$$\frac{c_l}{2} = Z \left(\frac{1}{M_\infty}\right)^e \left(\frac{1}{Re_c}\right)^f \tag{3.1}$$

Then we see that:

$$L = \frac{1}{2}\rho_\infty V_\infty^2 S c_l = q_\infty S c_l \tag{3.2}$$

And now we see that the lift coefficient is also equal to:

$$c_l = \frac{L}{q_\infty S} \tag{3.3}$$

Doing the same steps for the drag and the moment, will give:

$$D = q_\infty S c_d \tag{3.4}$$

$$M = q_\infty S c\, c_m \tag{3.5}$$

And finally we summarize all the equations:

$$c_l = \frac{L}{q_\infty S} \qquad c_d = \frac{D}{q_\infty S} \qquad c_m = \frac{M}{q_\infty S c} \tag{3.6}$$

$$c_l = f_1(\alpha, M_\infty, Re) \qquad c_d = f_2(\alpha, M_\infty, Re) \qquad c_m = f_3(\alpha, M_\infty, Re) \tag{3.7}$$

Where f_1, f_2 and f_3 are functions. This is to emphasize that the coefficients depend on the parameters noted in brackets.

4 Bending coefficients

As was already mentioned in the previous paragraph, the moment coefficient is defined as:

$$c_{m_x} = \frac{M_x}{q_\infty S c} \tag{4.1}$$

Where x can be any distance from the leading edge of the wing (so $0 \leq x \leq c$) and M is the moment acting on that point. Note that there is now an extra variable c in the equation, where there was none in the definition for the force coefficients. If the chord c wouldn't be present, c_m wouldn't be dimensionless.

There are two specific points concerning moments which are often used in aerodynamics. The first one is the center of pressure (dp in short). This is where there is no bending moment. The position of this point usually changes if the angle of attack changes. The second point is the aerodynamic center (ac). This is the point where the bending moment stays constant as the angle of attack changes. Its position, which is almost always around the quarter chord position ($\frac{x_{ac}}{c} \approx 0.25$), doesn't change if the angle of attack changes. So in formula:

$$c_{m_{dp}} = 0 \qquad \frac{dc_{m_{ac}}}{d\alpha} = 0 \tag{4.2}$$

So the moment coefficient is different on different positions on an airfoil. But there exists a relationship between those moment coefficients. And, using simple statics, it can be shown that:

$$c_{m_{Q_1}} - c_{m_{Q_2}} = c_n \left(\frac{x_{Q_1}}{c} - \frac{x_{Q_2}}{c} \right) \tag{4.3}$$

Do remember that this formula only applies for constant angle of attack, since otherwise variables like c_n change. However, the fact that $\frac{x_{ac}}{c}$ remains constant for different angles of attack can be used.

5 Prandtl-Glauert Rule

Let's define a new dimensionless coefficient to indicate the pressure over a wing. We define the pressure coefficient C_p as follows:

$$C_p = \frac{p - p_\infty}{q_\infty} = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} \tag{5.1}$$

The pressure coefficient can be plotted for a certain airfoil. Suppose the pressure coefficient at a point on an airfoil at low speeds ($M_\infty \approx 0$) is measured. If the air velocity increases, also the absolute value of the pressure coefficient increases (negative C_p get even more negative). This is, according to the Prandtl-Glauert rule, approximately equal to:

$$C_p = \frac{C_{p,0}}{\sqrt{1 - M_\infty^2}} \tag{5.2}$$

Where $C_{p,0}$ is the pressure coefficient at low speeds. For $M_\infty < 0.3$ compressibility effects don't need to be taken into account, and for $M_\infty > 0.7$ this formula loses its accuracy. Therefore this formula is only really applicable for $0.3 < M_\infty < 0.7$.

Now let's define the normal force coefficient and the axial force coefficient for unit length (for just $1m$ of the wing) in the same way as the lift coefficient:

$$c_n = \frac{N}{q_\infty c} \qquad c_a = \frac{A}{q_\infty c} \tag{5.3}$$

It can then be shown that the normal force coefficient is also equal to the following integral:

$$c_n = \int_0^1 (C_{p,l} - C_{p,u}) d\frac{x}{c} \tag{5.4}$$

Combining this with equation 2.1 results in:

$$c_l = c_n \cos\alpha - c_a \sin\alpha \tag{5.5}$$

Most aircrafts have their cruising angle of attack at $\alpha < 5°$. And for such small angles of attack, $\sin\alpha \to 0$ and $\cos\alpha \to 1$. So $c_l \approx c_n$, and then equation 5.4 can also be used to calculate the lift coefficient.

But for this lift coefficient, the Prandtl-Glauert rule can also be applied. If $c_{l,0}$ is the lift coefficient for low air velocities ($M_\infty < 0.3$), then the lift coefficient at higher Mach numbers is:

$$c_l = \frac{c_{l,0}}{\sqrt{1 - M_\infty^2}} \tag{5.6}$$

6 Critical pressure coefficient

Using the definition of the pressure coefficient, the definition of the dynamic pressure, the formula for the speed of sound, the isentropic flow relations and a few assumptions, the following formula can be derived:

$$C_p = \frac{2}{\gamma M_\infty^2}\left(\left[\frac{1+\frac{1}{2}(\gamma-1)M_\infty^2}{1+\frac{1}{2}(\gamma-1)M^2}\right]^{\frac{\gamma}{\gamma-1}} - 1\right) \tag{6.1}$$

So for a certain atmosphere and a constant free-stream Mach number, the pressure coefficient only depends on the local Mach number M on the wing. To find the critical pressure coefficient, we should fill in $M = 1$. This results in:

$$C_p = \frac{2}{\gamma M_\infty^2}\left(\left[\frac{2+(\gamma-1)M_\infty^2}{\gamma+1}\right]^{\frac{\gamma}{\gamma-1}} - 1\right) \tag{6.2}$$

So the critical pressure coefficient only depends on the Mach number, which is quite an interesting thing. To find the critical pressure coefficient, the Prandtl-Glauert rule should be used. If M_{cr} is the critical Mach number, and $C_{p,0}$ is the lowest pressure coefficient on the wing for low air velocities, then the following formula applies:

$$\frac{C_{p,0}}{\sqrt{1-M_{cr}^2}} = C_p = \frac{2}{\gamma M_{cr}^2}\left(\left[\frac{2+(\gamma-1)M_{cr}^2}{\gamma+1}\right]^{\frac{\gamma}{\gamma-1}} - 1\right) \tag{6.3}$$

7 Supersonic flight

To reduce the critical Mach number M_{cr}, a plane can be equipped with swept wings. Because these wings have a certain angle with respect to the airflow, the speed of the air relative to the leading edge of the wing is reduced. If Ω is the angle between the leading edge of the wing, and the line perpendicular to the direction of flight (so for normal planes $\Omega = 0$), then the new critical Mach number is $\frac{M_{cr}}{\cos\Omega}$. However, to use the component of the air velocity normal to the leading edge of the swept wings is not always accurate. However, the following formula is correct:

$$M_{cr,normal} < M_{cr,swept} < \frac{M_{cr,normal}}{\cos\Omega} \tag{7.1}$$

But if supersonic flight does happen, shock waves occur. These shock waves have a certain angle with respect to the direction of flight. For flat plates, this angle can easily be calculated:

$$\mu = \arcsin\frac{1}{M} \tag{7.2}$$