Basics of Inviscid Incompressible Flows

1 Bernoulli's equation

An **incompressible flow** is a flow where the density ρ is constant. Let's assume we're dealing with an incompressible flow. From the momentum equation and the streamline condition, we can derive that

$$dp = -\rho V \, dV. \tag{1.1}$$

This equation is called **Euler's equation**. Since the streamline condition was used in the derivation, it is only valid along a streamline. Integrating the Euler equation between point 1 and point 2 gives

$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2. \tag{1.2}$$

In other words, $p + \frac{1}{2}\rho V^2$ is constant along a streamline.

An **inviscid** flow is a flow without friction, thermal conduction or diffusion. It can be shown that inviscid flows are irrotational flows. For irrotational flows $p + \frac{1}{2}\rho V^2$ is constant, even for different streamlines.

2 Continuity equation

In a low-speed wind tunnel the flow field variables can be assumed to be a function of x only, so $A = A(x)$, $V = V(x)$, $p = p(x)$, etcetera. Such a flow is called a **quasi-one-dimensional flow**. From the continuity equation can be derived that

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2, \tag{2.1}$$

for two points in the tunnel. This applies to both compressible and incompressible flows. If the flow becomes incompressible, then $\rho_1 = \rho_2$. The equation then reduces to $A_1 V_1 = A_2 V_2$. If we combine this with Bernoulli's equation, we find

$$V_1 = \sqrt{\frac{2(p_1 - p_2)}{\rho\left(\frac{A_1}{A_2} - 1\right)}}. \tag{2.2}$$

3 Dynamic pressure

The **dynamic pressure** is defined as

$$q = \frac{1}{2}\rho V^2. \tag{3.1}$$

Let's suppose that the velocity at some point 0 is zero ($V_0 = 0$). If the flow is incompressible, it follows that

$$p_1 + \frac{1}{2}\rho V_1^2 = p_0 \qquad \Rightarrow \qquad q_1 = p_0 - p_1. \tag{3.2}$$

Note that this follows from Bernoulli's equation. If the flow is compressible, Bernoulli's equation is not valid and thus $p_0 - p_1 \neq q_1$.

4 Pressure coefficient

The **pressure coefficient** C_p is defined as

$$C_p = \frac{p - p_\infty}{q_\infty}, \tag{4.1}$$

where $q_\infty = \frac{1}{2}\rho_\infty V_\infty^2$. The ∞ subscript denotes that the values are measured in the free stream, as if being infinitely far away from the examined object. For incompressible flows, C_p can also be written as

$$C_p = 1 - \left(\frac{V}{V_\infty}\right)^2. \tag{4.2}$$

5 Laplace's equation

If the flow is incompressible, it follows from the continuity equation that

$$\nabla \cdot \mathbf{V} = 0. \tag{5.1}$$

If the flow is also inviscid, and thus irrotational, it follows that $\nabla \times \mathbf{V} = 0$. It also implicates that there is a velocity potential ϕ such that $\mathbf{V} = \nabla\phi$. Combining this with equation 5.1 gives

$$\nabla \cdot (\nabla\phi) = \nabla^2\phi = 0. \tag{5.2}$$

This simple but important relation is called **Laplace's equation**. It seems that the velocity potential satisfies Laplace's equation. But what about the stream function? We can recall from the previous chapter that

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}. \tag{5.3}$$

We can also remember the irrotationality condition, stating that $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$. Inserting 5.3 in this condition gives

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \qquad \Rightarrow \qquad \nabla^2\psi = 0. \tag{5.4}$$

So the stream function ψ also satisfies Laplace's equation, just like the velocity potential function ϕ.

6 Applying Laplace's equation

Note that the Laplace equation is a linear partial differential equation. So if we find multiple solutions $\phi_1, \ldots, \phi_n$ for it, then any linear combination $\phi = c_1\phi_1 + \ldots + c_n\phi_n$ is also a solution. So if we find a couple of basic solutions to Laplace's equation, and if we add them up in just the right way, we can display any inviscid incompressible flow.

But how do we know how to put the independent solutions together? We have to make use of **boundary conditions**. First, there are the **boundary conditions on velocity at infinity**, stating that, at infinity,

$$u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} = V_\infty, \qquad v = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} = 0. \tag{6.1}$$

There are also the **wall boundary conditions**. The flow can not penetrate an airfoil. So the velocity at the airfoil edge is directed tangentially. This can be expressed in many ways. If $\mathbf{n}$ is the normal vector at the airfoil surface, then $\mathbf{V} \cdot \mathbf{n} = (\nabla\phi) \cdot \mathbf{n} = 0$. This is called the **flow tangency condition**. But since the airfoil edge is a streamline itself, also $\psi_{surface} =$ constant.

If we are dealing with neither ϕ or ψ, but rather with u and v themselves, things are different. If the shape of the airfoil is given by $y_b(x)$, then

$$\frac{dy_b}{dx} = \left(\frac{v}{u}\right)_{surface}. \tag{6.2}$$

With those boundary conditions, we can put the elementary solutions to Laplace's equation together to represent, for example, the flow over a cylinder or over an airfoil. All that is left now, is to find those elementary solutions. That is the subject of the next chapter.