

Basic Concepts

In this summary we will examine compressible flows. But before we venture into the depths of the aerodynamics, we will examine some basic concepts.

1 Basic Concepts of Gases

Usually the atoms in a gas exert forces on each other. If these **intermolecular forces** are negligible, we are dealing with a **perfect gas**. For perfect gases the following **equation of state** is applicable:

$$p = \rho RT, \quad (1.1)$$

where p is the **pressure**, ρ is the **density** and T is the **temperature**. R is the **specific gas constant**. Its value is $R = 287 \text{ J/kg K}$ at standard sea-level conditions.

Every molecule in a gas has a certain amount of energy. The sum of all these energies is called the **internal energy** of the gas. The internal energy per unit mass is called the **specific internal energy** e . There also is the **specific enthalpy** h , defined as

$$h = e + pv, \quad (1.2)$$

where $v = 1/\rho$ is the **specific volume**. For a perfect gas, both e and h are functions of only the temperature T . In fact, we have

$$de = c_v dT \quad \text{and} \quad dh = c_p dT, \quad (1.3)$$

where c_v and c_p are the **specific heat at constant volume** and **specific heat at constant pressure**, respectively. Often c_v and c_p also depend on the temperature T . If they can be assumed constant, then the gas is called a **calorically perfect gas**. We then have

$$e = c_v T \quad \text{and} \quad h = c_p T. \quad (1.4)$$

Let's take a closer look at the variables c_v , c_p and R . There are relations between them. If we also define $\gamma = c_p/c_v$, then it can be shown that

$$\gamma = \frac{c_p}{c_v}, \quad R = c_p - c_v, \quad (1.5)$$

$$c_p = \frac{\gamma R}{\gamma - 1}, \quad c_v = \frac{R}{\gamma - 1}. \quad (1.6)$$

2 The First Law of Thermodynamics

Let's consider a fixed mass of gas, called the **system**. The region outside the system is called the **surroundings**. In between the surroundings and the system is the **boundary**. We can now state the **first law of thermodynamics**, being

$$de = \delta q + \delta w. \quad (2.1)$$

Here δq is the amount of heat added and δw is the amount of work done on the system.

Heat can be added and work can be done in many ways. In **adiabatic processes** no heat is added or taken away from the system. In **reversible processes** things like mass diffusion, viscosity and thermal conductivity are absent. Finally **isentropic processes** are both adiabatic and reversible.

3 The Second Law of Thermodynamics

It is time to define the **entropy** s of a system. The **second law of thermodynamics** states that

$$ds \geq \frac{\delta q}{T}, \quad (3.1)$$

where there is only equality for reversible processes. Furthermore, if the process is adiabatic, then $\delta q = 0$ and thus also

$$ds \geq 0. \quad (3.2)$$

If the process is both reversible and adiabatic, then $ds = 0$. The entropy is thus constant for isentropic processes. (This also explains why these processes were named isentropic.)

Now let's try to derive an equation for the entropy. We do this using the first law of thermodynamics. For a reversible process it can be shown that $\delta w = -p dv$. Also we have $\delta q = T ds$. From this we can find that

$$T ds = de + p dv = dh - v dp. \quad (3.3)$$

We can combine the above relations with the equation of state and the relations for de and dh . Doing this will eventually result in

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} = c_v \ln \frac{T_2}{T_1} + R \ln \frac{v_2}{v_1}. \quad (3.4)$$

For isentropic processes we have $ds = 0$ and thus $s_2 - s_1 = 0$. Using this fact, we can find the **isentropic flow relations**, being

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1} \right)^\gamma = \left(\frac{T_2}{T_1} \right)^{\frac{\gamma}{\gamma-1}}. \quad (3.5)$$

4 Compressibility

Let's consider some substance. If we increase the pressure on it, its volume will decrease. We can now define the **compressibility** τ as

$$\tau = -\frac{1}{v} \frac{dv}{dp}. \quad (4.1)$$

However, when the pressure is increased often also the temperature and the entropy increase. To erase these effects, we define the **isothermal compressibility** τ_T and the **isentropic compressibility** τ_s as the compressibility at isothermal and isentropic processes, respectively. In an equation, this becomes

$$\tau_T = -\frac{1}{v} \left(\frac{\partial v}{\partial p} \right)_T \quad \text{and} \quad \tau_s = -\frac{1}{v} \left(\frac{\partial v}{\partial p} \right)_s. \quad (4.2)$$

But how can we use this? Using $v = 1/\rho$ we can derive that

$$d\rho = \rho \tau dp. \quad (4.3)$$

This equation helps us to judge whether a flow is compressible. A flow is **incompressible** when the density stays (more or less) constant throughout the process. If the density varies, then the flow is compressible. For low-speed flows dp is small, so also $d\rho$ is small. The flow is thus incompressible. For high-speed flows the pressure will change a lot more. Therefore $d\rho$ is not small anymore, and the flow is thus compressible.

5 Stagnation Conditions

Let's consider a flow with velocity V . If we move along with the flow, we can measure a certain static pressure p . We can also measure the density ρ , the temperature T , the Mach number M , and so on. All these quantities are static quantities.

Now let's suppose we slow down the flow adiabatically to $V = 0$. The temperature, pressure and density of the flow now change. The new value of the temperature is defined as the **total temperature** T_t . The corresponding **total enthalpy** is $h_t = c_p T_t$.

Using a rather lengthy derivation, it can be shown that the quantity $h + V^2/2$ stays constant along a streamline, in a steady adiabatic inviscid flow. We therefore have

$$h_t = h + \frac{V^2}{2} = \text{constant}. \quad (5.1)$$

For a calorically perfect gas (with constant c_p) we also have $T_t = h_t/c_p = \text{constant}$. Keep in mind that this only holds for adiabatic flows.

We can expand this idea even further, if the flow is also reversible, and thus isentropic. In this case, it turns out that the **total pressure** p_t and the **total density** ρ_t also stay constant along a streamline.