Basic Applications of Elementary Flows

1 Nonlifting flow over a cylinder

If we combine a uniform flow with a doublet, we get the stream function

$$\psi = V_\infty r \sin\theta - \frac{\kappa}{2\pi}\frac{\sin\theta}{r} = V_\infty r \sin\theta \left(1 - \frac{\kappa}{2\pi V_\infty r^2}\right) = V_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right), \tag{1.1}$$

where $R^2 = \frac{\kappa}{2\pi V_\infty}$. This is also the stream function for a flow over a cylinder/circle with radius

$$R = \sqrt{\frac{\kappa}{2\pi V_\infty}}. \tag{1.2}$$

The velocity field can be found by using the stream function, and is given by

$$V_r = V_\infty \cos\theta \left(1 - \frac{R^2}{r^2}\right), \qquad V_\theta = -V_\infty \sin\theta \left(1 + \frac{R^2}{r^2}\right). \tag{1.3}$$

Note that if $r = R$, then $V_r = 0$, satisfying the wall boundary condition. At the wall also $V_\theta = -2V_\infty \sin\theta$. This means that the pressure coefficient over the cylinder is given by

$$C_p = 1 - \left(\frac{V}{V_\infty}\right)^2 = 1 - 4\sin^2\theta. \tag{1.4}$$

2 Nonlifting flow over a sphere

Let's combine a uniform 3-dimensional flow with a 3-dimensional doublet. Let's define R as

$$R = \sqrt[3]{\frac{\mu}{2\pi V_\infty}}. \tag{2.1}$$

Using the combined stream function, it can be shown that the velocity field is given by

$$V_r = -V_\infty \cos\theta \left(1 - \frac{R^3}{r^3}\right), \qquad V_\theta = V_\infty \sin\theta \left(1 + \frac{R^3}{r^3}\right), \qquad V_\phi = 0. \tag{2.2}$$

At the wall, the velocity is $V_\theta = \frac{3}{2}V_\infty \sin\theta$. This means that the pressure coefficient over the sphere is given by

$$C_p = 1 - \left(\frac{V}{V_\infty}\right)^2 = 1 - \frac{9}{4}\sin^2\theta. \tag{2.3}$$

3 Lifting flow over a cylinder

Let's combine a nonlifting flow over a cylinder with a vortex of strength Γ. This results in a lifting flow over a cylinder. The resulting stream function is

$$\psi = (V_\infty r \sin\theta)\left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\ln\frac{r}{R}. \tag{3.1}$$

From the stream function we can derive the velocity field, which is given by

$$V_r = V_\infty \cos\theta \left(1 - \frac{R^2}{r^2}\right), \qquad V_\theta = -V_\infty \sin\theta \left(1 + \frac{R^2}{r^2}\right) - \frac{\Gamma}{2\pi r}. \tag{3.2}$$

To find the stagnation points, we simply have to set V_r and V_θ to 0. If $\frac{\Gamma}{4\pi V_\infty R} \leq 1$, then the solution is given by

$$r = R, \qquad \theta = \arcsin\left(-\frac{\Gamma}{4\pi V_\infty R}\right). \tag{3.3}$$

However, if $\frac{\Gamma}{4\pi V_\infty R} \geq 1$, then the solution is given by

$$r = \frac{\Gamma}{4\pi V_\infty} \pm \sqrt{\left(\frac{\Gamma}{4\pi V_\infty}\right)^2 - R^2}, \qquad \theta = -\frac{1}{2}\pi. \tag{3.4}$$

At the surface of the cylinder (where $r = R$) is the velocity $V = V_\theta$. Using this, the pressure coefficient can be calculated. The result is

$$C_p = 1 - \left(2\sin\theta + \frac{\Gamma}{2\pi R V_\infty}\right)^2. \tag{3.5}$$

Using this pressure coefficient, the drag coefficient can be found to be $c_d = 0$. So there is no drag. Also, the lift coefficient is

$$c_l = \frac{\Gamma}{R V_\infty}, \tag{3.6}$$

where $R = \frac{1}{2}c$. Now the lift per unit span L' can be obtained from

$$L' = c_l q_\infty c = \frac{\Gamma}{R V_\infty}\frac{1}{2}\rho_\infty V_\infty^2 2R = \rho_\infty V_\infty \gamma. \tag{3.7}$$

This equation is called the **Kutta-Joukowski Theorem**. It states that the lift per unit span is directly proportional to the circulation. It also works for shapes other than cylinders. However, for other shapes a complex distribution of sources and vortices may be necessary, as is the subject of the following paragraph.

4 Source Panel Method

The **source panel technique** is a numerical method to use elementary flows. Let's put a lot of sources along a curve with source strength per unit length $\lambda = \lambda(s)$. Such a source distribution is called a **source sheet**. Note that λ can be positive at some points and negative in other points.

Now look at an infinitely small part of the source sheet. The source strength of this part is $\lambda\, ds$. So for any point P, the contribution of this small source sheet part to the velocity potential is

$$d\phi = \frac{\lambda\, ds}{2\pi}\ln r, \tag{4.1}$$

where r is the distance between the source sheet part and point P. The entire velocity potential can be obtained by integrating, which simply gives

$$\phi = \int_a^b \frac{\lambda\, ds}{2\pi}\ln r. \tag{4.2}$$

In the source panel method, usually an airfoil (or an other shape) is split up in a number of small straight lines for which the velocity potential is separately calculated and the boundary conditions are separately applied.