Aerodynamics Lines and Equations

1 Pathlines, streamlines and streaklines

A **pathline** is a curve in space traced out by a certain particle in time. A **streamline** is a line where the flow is tangential. A **streakline** is the line formed by all the particles that previously passed through a certain point. However, for steady flows, pathlines, streamlines and streaklines simply coincide.

Of these three lines, streamlines are the lines most used in aerodynamics. But how can we find the equation for a streamline? Since the velocity vector is tangential to the streamline at that point, we know that

$$\mathbf{V} \times \mathbf{ds} = \mathbf{0}. \tag{1.1}$$

Looking at the components of the vectors also shows that

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \tag{1.2}$$

Note that for 2-dimensional situations this equation reduces to

$$v\,dx = u\,dy. \tag{1.3}$$

2 Vorticity

If ω is the angular velocity of a small volume in space, then the **vorticity** ξ is defined as

$$\xi = 2\omega = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k} = \nabla \times \mathbf{V}. \tag{2.1}$$

So in a velocity field, the curl of the velocity is equal to the vorticity. If $\xi = 0$ at every point in a flow, the flow is called **irrotational**. The motion of fluid elements is then without rotation - there is only pure translation. If $\xi \neq 0$ for some point, then the flow is called **rotational**.

Note that for 2-dimensional flows the vorticity is given by $\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. So if the flow is irrotational, then

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}. \tag{2.2}$$

This equation is the **condition of irrotationality for two-dimensional flow** and will be used quite frequently.

3 Circulation

Consider a closed curve C in a flow field. The **circulation** Γ is defined as

$$\Gamma = -\oint_C \mathbf{V} \cdot \mathbf{ds}. \tag{3.1}$$

By definition the integral along C has counter-clockwise as positive direction, but by definition the circulation has clockwise as positive direction. Therefore the minus sign is present in the equation.

If the flow is irrotational everywhere in the surface bounded by C, then $\Gamma = 0$.

4 Stream functions

Let's consider a 2-dimensional flow for now. If the velocity distribution of the flow is known, equation 1.3 can be integrated to find the equation for a streamline $\bar{\psi}(x, y) = c$. The function $\bar{\psi}$ is called the **stream function**. Different values of c result in different streamlines.

If a stream function ψ is known, then the product ρV at a certain point in the flow can be found, using

$$\rho u = \frac{\partial \bar{\psi}}{\partial y}, \qquad \rho v = -\frac{\partial \bar{\psi}}{\partial x}. \tag{4.1}$$

Now suppose we're dealing with incompressible flows, and thus $\rho = \text{constant}$. Let's define a new stream function $\psi = \bar{\psi}/\rho$. (Note that ψ has unit $[m^3/(s\,m) = m^2/s]$.) Then equation 4.1 becomes

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{4.2}$$

In polar coordinates this becomes

$$V_r = \frac{1}{r}\frac{d\psi}{d\theta}, \qquad V_\theta = -\frac{d\psi}{dr}. \tag{4.3}$$

5 Velocity potential

For an irrotational flow, it is known that $\xi = \nabla \times \mathbf{V} = 0$. There is also a vector identity, stating that $\nabla \times (\nabla \phi) = 0$. Combining these equations, we see that there is a scalar function ϕ such that.

$$\mathbf{V} = \nabla \phi. \tag{5.1}$$

The function ϕ is called the **velocity potential**. If the velocity potential is known, then the velocity at every point can be determined, using

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}, \qquad w = \frac{\partial \phi}{\partial z}. \tag{5.2}$$

In polar coordinates, this is

$$V_r = \frac{d\phi}{dr}, \qquad V_\theta = \frac{1}{r}\frac{d\phi}{d\theta}. \tag{5.3}$$

Since irrotational flows can be described by a velocity potential ϕ, such flows are also called **potential flows**.

6 Stream function versus velocity potential

The stream function and the velocity potential have important similarities and differences. Keep in mind that the velocity potential is defined for irrotational flow only, while the stream function can be used for both rotational and irrotational flows. On the contrary, the velocity potential applies for three-dimensional flows, while the stream function is defined for two-dimensional flows only.

There is another interesting relation between the stream function and the velocity potential. Suppose we plot lines for constant values of the stream function $\psi = \text{constant}$. The streamlines do not intersect other stream lines. Now we also plot lines for constant values of the velocity potential $\phi = \text{constant}$, being so-called **equipotential lines**. The equipotential lines do not intersect other equipotential lines either. However, the streamlines and the equipotential lines do intersect. The peculiar thing is that they always intersect perpendicular. This, in fact, can be mathematically proven. So streamlines and equipotential lines are orthogonal.